

§ Euler-Lagrange flows

$L:TM \rightarrow R$ Lagrangian , $u:[0,1] \rightarrow M$

$$A(u) = \int_0^1 L(u(t), \dot{u}(t)) dt$$

$$\frac{\partial L}{\partial u}(u, \dot{u}) - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{u}}(u, \dot{u}) \right) = 0$$

If M is compact , the extremals of A give rise to a complete flow $\phi_t: TM \rightarrow TM$ called the Euler-Lagrange flow of the Lagrangian .

[DG12]Geometric Analysis p.23

Lemma

The Euler-Lagrange equations for the energy E are

$$\ddot{x}^i(t) + \Gamma_{jk}^i(x(t)) \dot{x}^j(t) \dot{x}^k(t) = 0$$

$$\text{Where } \Gamma_{jk}^i = \frac{1}{2} g^{il} (g_{jl,k} + g_{kl,j} - g_{jk,l})$$

1. For $I(x) = \int_a^b f(x(t), \dot{x}(t), t) dt$ then the E-L equation are $\frac{\partial f}{\partial x^i} - \frac{d}{dt} \left(\frac{\partial f}{\partial \dot{x}^i} \right) = 0$

$$2. E(\gamma) = \frac{1}{2} \int g_{jk}(x(t)) \dot{x}^j \dot{x}^k dt$$

Note that $\frac{\partial}{\partial x^i} (g_{jk} \dot{x}^j \dot{x}^k) = g_{ik} \dot{x}^k + g_{ji} \dot{x}^j$ 因為 $\frac{\partial}{\partial x^i} \dot{x}^j = \delta_{ij}$

The E-L equations are $\frac{\partial}{\partial x^i} (g_{jk} \dot{x}^j \dot{x}^k) - \frac{d}{dt} \left(\frac{\partial}{\partial \dot{x}^i} g_{jk} \dot{x}^j \dot{x}^k \right) = 0$

$$\frac{\partial g_{jk}}{\partial x^i} \dot{x}^j \dot{x}^k - \left\{ \frac{\partial g_{ik}}{\partial x^l} \frac{\partial x^l}{\partial t} \dot{x}^k + \frac{\partial g_{jl}}{\partial x^l} \frac{\partial x^l}{\partial t} \dot{x}^j + g_{ik} \ddot{x}^k + g_{ji} \ddot{x}^j \right\} = 0$$

$$g_{jk,i} \dot{x}^j \dot{x}^k - \left\{ g_{ik,l} \dot{x}^l \dot{x}^k + g_{ji,l} \dot{x}^l \dot{x}^j + g_{ik} \ddot{x}^k + g_{ji} \ddot{x}^j \right\} = 0$$

Renaming some indices and using the symmetry $g_{ik} = g_{ki}$, we get

$$2g_{\ell m} \ddot{x}^m + (g_{\ell k,j} + g_{j\ell,k} - g_{jk,\ell}) \dot{x}^j \dot{x}^k = 0, \quad \ell = 1, \dots, d,$$

And from this $g^{il} g_{lm} \ddot{x}^m + \frac{1}{2} g^{il} (g_{lk,j} + g_{jl,k} - g_{jk,l}) \dot{x}^j \dot{x}^k = 0$

$g^{il} g_{lm} = \delta_{im}$, so $g^{il} g_{lm} \ddot{x}^m = \ddot{x}^i$, that is $\ddot{x}^i + \Gamma_{jk}^i \dot{x}^j \dot{x}^k = 0$

§ Cartan magic formula $L_X \omega = \iota_X d\omega + d(\iota_X \omega)$

右式前者是 interior product(interior derivative) , 後者是 exterior derivative

$$X \in \chi(M) , L_X \omega = \frac{d}{dt} (\varphi_t^* \omega) \Big|_{t=0}$$

$$L_X d\omega = dL_X \omega$$

$$L_X (\omega \wedge \eta) = L_X \omega \wedge \eta + \omega \wedge L_X \eta$$

1. 例子 in RG1102

$$X = F \frac{\partial}{\partial x} + G \frac{\partial}{\partial y} + H \frac{\partial}{\partial z} , \text{ volume form } dv = dx \wedge dy \wedge dz$$

$$L_X dx = d(L_X x) = d(Xx) = dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy + \frac{\partial F}{\partial z} dz , \text{ Then}$$

$$\begin{aligned} L_X dv &= L_X (dx \wedge dy \wedge dz) = (L_X dx) \wedge dy \wedge dz + dx \wedge (L_X dy) \wedge dz + dx \wedge dy \wedge (L_X dz) \\ &= \left(\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z} \right) dv = (div X) dv \end{aligned}$$

驗證 所謂的 interior product 如何運算。

$$\iota_X (dx_1 \wedge dx_2 \dots \wedge dx_n) = \sum_{r=1}^n (-1)^{r-1} f_r dx_1 \wedge \dots \wedge \hat{dx_r} \wedge \dots \wedge dx_n$$

這裡 $\hat{dx_r}$ 表示把 dx_r 省略

$$\text{所以 } X = F \frac{\partial}{\partial x} + G \frac{\partial}{\partial y} + H \frac{\partial}{\partial z} = (F, G, H) , \omega = dx \wedge dy \wedge dz$$

$$\iota_X \omega = F dy \wedge dz - G dx \wedge dz + H dx \wedge dy$$

$$d(\iota_X \omega) = dF \wedge dy \wedge dz - dG \wedge dx \wedge dz + dH \wedge dx \wedge dy$$

$$\text{其中 } dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy + \frac{\partial F}{\partial z} dz , \dots$$

$$\text{所以 } d(\iota_X \omega) = \left(\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} + \frac{\partial H}{\partial z} \right) dx \wedge dy \wedge dz$$