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Generating Inequalities and Soliton Solutions via AI-Assisted Symbolic Computation

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Abstract

This paper explores the application of artificial intelligence, specifically the DeepSeek model, in generating mathematical inequalities and constructing soliton solutions to integrable systems. We demonstrate two primary applications: first, the automated derivation of inequalities such as the JM inequality from matrix power means, leading to classical results like Minkowski's inequality; second, the use of auto-Bäcklund and Darboux transformations to generate new soliton solutions to the sine-Gordon equation from trivial seeds. We compare the performance of several AI models—DeepSeek, ChatGPT, Perplexity, and Gemini—in solving the same Bäcklund transformation problem. The process illustrates how AI can perform symbolic computations that traditionally require deep mathematical insight, offering a reproducible and accessible approach to exploring nonlinear systems and inequalities. This work highlights the potential of AI as a collaborative tool in mathematical research, especially in the age of widely accessible computational intelligence.

Mathematical AI, Soliton Solutions, Bäcklund Transformation, Inequality Generation, Integrable Systems, Sine-Gordon Equation

1 Introduction

Recent advances in artificial intelligence have opened new avenues for mathematical exploration. The release of DeepSeek in January 2025 marks a significant milestone in making advanced AI tools accessible to researchers across disciplines. In this paper, we illustrate how DeepSeek and other AI models can be used to generate meaningful mathematical results in two distinct areas: inequality derivation and soliton theory.

We show that with carefully designed prompts, these models can reproduce known results and assist in constructing new solutions to nonlinear partial differential equations. This approach demonstrates the potential of AI as a collaborative partner in mathematical discovery, particularly in fields requiring sophisticated symbolic computation.

2 Generating Inequalities via Power Means

Let $A = (a_{ij})$ be an $m \times n$ matrix of positive real numbers. For $p \in R$, the power mean M_p is defined as:

$$M_p(a_1, \dots, a_n) = \{ \ (\frac{1}{n}\sum_{i=1}^{n} a_i^{p1/p}, p \neq 0 \left(\prod_{i=1}^{n} a_i\right)^{1/n}, p = 0$$

The JM inequality, introduced by Lin et al. in their work on matrices and inequalities [1], states that for $p > q$,

$$M_q\left(\left(M_p((a_{ij})_{j=1}^{n}\right)_{i=1}^{m}\right) \geq M_p\left(\left(M_q((a_{ij})_{i=1}^{m}\right)_{j=1}^{n}\right).$$

For example, with $A = ab$
cd, $p = 2$, $q = 1$, the JM inequality yields:

$$\sqrt{\frac{a^2 + b^2 + c^2 + d^2}{2}} \geq \sqrt{\frac{(a+c)^2 + (b+d)^2}{2}},$$

which is equivalent to the well-known Minkowski inequality [2]:

$$(\sum_{k=1}^{n} |a_k + b_k|^p)^{1/p} \leq (\sum_{k=1}^{n} |a_k|^p)^{1/p} + (\sum_{k=1}^{n} |b_k|^p)^{1/p}, \quad p \geq 1.$$

This example shows how AI can be prompted to generate non-trivial inequalities from general principles, providing an automated approach to inequality discovery.

3 Generating Soliton Solutions

3.1 Bäcklund Transformation for the Sine-Gordon Equation

The sine-Gordon equation is a well-known integrable system with applications in differential geometry and physics. Its auto-Bäcklund transformation is given by:

$$\{ \ \partial_{\xi}\,(\phi_1 - \phi_0) = 2\lambda \sin\left(\frac{\phi_1 + \phi_0}{2}\right), \partial_\eta(\phi_1 + \phi_0) = \frac{2}{\lambda}\sin\left(\frac{\phi_1 - \phi_0}{2}\right),$$

where λ is a spectral parameter. Starting from the trivial solution $\phi_0 = 0$, we prompted several AI models to solve for ϕ_1. The results are summarized in Table 1.

DeepSeek provided the most standard and directly usable form of the kink solution. ChatGPT introduced an unnecessary constant K, while Perplexity reformulated the solution in physical variables, showing interpretative flexibility. Gemini was unable to process the problem effectively.

Table 1: Comparison of AI models in solving the Bäcklund transformation

Model	**Solution**
DeepSeek	$\phi_1(\xi,\eta) = 4\arctan\left(e^{\lambda\xi+\eta/\lambda}\right)$
ChatGPT	$\phi_1(\xi,\eta) = 4\arctan\left(Ke^{\lambda\xi+\eta/\lambda}\right)$, where K is an arbitrary constant
Perplexity	$\phi_1(x,t) = 4\arctan\left(e^{x-\nu t/\sqrt{1-\nu^2}}\right)$, expressed in physical coordinates
Gemini	Failed to recognize the problem structure or provide a valid solution

3.2 Darboux Transformation and Multi-Soliton Solutions

The Darboux transformation provides an algebraic method for constructing new solutions to integrable systems. For the sine-Gordon equation with Lax pair:

$$\partial_x \Psi = U\Psi, \quad \partial_y \Psi = V\Psi,$$

where

$$\Psi = (\,\psi\,)_1\,\psi_2, \quad U = \frac{i}{2}\,(\,\partial\,)_t\,\phi\lambda\lambda - \partial_t\phi, \quad V = (\,0\,)\,e^{i\phi}e^{-i\phi}0,$$

starting from the trivial solution $\phi_0 = 0$, DeepSeek generated the same kink solution via Darboux transformation.

Using the kink solution as a seed, a two-soliton solution was also generated via the Bäcklund transformation:

$$\phi_2 = 4\arctan\left(\frac{\lambda+\mu}{\lambda-\mu}\cdot\frac{e^{\lambda\xi+\eta/\lambda} - e^{\mu\xi+\eta/\mu}}{1+e^{(\lambda+\mu)\xi+\left(\frac{1}{\lambda}+\frac{1}{\mu}\right)\eta}}\right).$$

For complex spectral parameters $\lambda = a + ib$, this approach can generate breather solutions, representing bound states of kink-antikink pairs.

3.3 KdV Equation Solutions

The Korteweg-de Vries (KdV) equation, another celebrated integrable system, also admits solution generation via Darboux transformation. Starting from the trivial solution $u_0 = 0$, the single-soliton solution is:

$$u_1 = -2k_1^2 sech^2(k_1x - 4k_1^3t).$$

Using u_1 as a seed solution and selecting a new spectral parameter λ_2, the two-soliton solution can be expressed as:

$$u_2 = u_1 - 2\partial_x^2 \ln W(\psi_1(\lambda_1), \psi_2(\lambda_2)),$$

where W denotes the Wronskian determinant. This method provides a unified framework for constructing solutions to integrable systems, demonstrating the deep connection between integrability and nonlinear phenomena.

4 Discussion and Conclusion

Our experiments demonstrate that AI models like DeepSeek can serve as effective assistants in mathematical discovery. They are capable of both algebraic manipulation and conceptual interpretation—such as recognizing generated inequalities as known results or constructing multi-soliton solutions from seed functions. However, the performance varies significantly across models, as seen in the Bäcklund transformation example.

The comparative analysis reveals that while some models like DeepSeek provide mathematically precise results, others may introduce unnecessary parameters or reformulate solutions in different coordinate systems. This variability highlights the importance of model selection and prompt engineering when using AI for mathematical research.

The year 2025 marks a new era in which AI becomes widely accessible to researchers and students alike. Its ability to perform symbolic computations and generate step-by-step derivations will likely accelerate exploration in fields ranging from analysis to integrable systems. As AI models continue to improve, we anticipate they will become increasingly valuable collaborators in mathematical discovery.

This work represents a preliminary step toward integrating AI into the daily practice of mathematical research. Future directions include developing more sophisticated prompting strategies, creating specialized mathematical AI assistants, and establishing benchmarks for evaluating AI performance on mathematical tasks.

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A Prompt Examples

A.1 JM Inequality Prompt

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"Given a set of positive real numbers \(a_1, a_2, ..., a_n\) and a real number p
the power mean \(M_p\) is defined as:
[definition provided]
A={a_{ij}}_{\text{mean}} is a matrix
if p>q, then \(M_q((M_p((a_{ij})_{j=1}^n))^m) \geq M_p((M_q((a_{ij})_{j=1}^n))^m)\)
is called the JM inequality.
Let \(A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, p=2, q=1\)
produce a inequality explicitly by JM inequality."
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A.2 Bäcklund Transformation Prompt

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"The auto-Bäcklund transformation for SG is:
[equations provided]
Let \(\phi_0 = 0\), find \(\phi_1\)"
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References

[1] Lin, F. L., and Lin, Z. Q. (2024). Matrices and Inequalities. Mathematical Communication Quarterly, 49(1).

[2] Hardy, G. H., Littlewood, J. E., and Pólya, G. (1952). Inequalities. Cambridge University Press.

[3] Rogers, C., and Schief, W. K. (2002). Bäcklund and Darboux Transformations: Geometry and Modern Applications in Soliton Theory. Cambridge University Press.

[4] Gu, C. H., and Hu, H. S. (2005). Darboux Transformations in Integrable Systems. Springer.