Prime Number Theorem

$$\pi(x) := \sum_{p \le x} 1 \sim \frac{x}{\log(x)} \quad \text{as} \quad x \to \infty$$

一、引言

質數猜想由高斯在 1792 年提出(他當時 15 歲...)，他猜測質數的累積分布在 x 趨近於無限大時，會近似於 $\frac{x}{A\log(x)+B}$ 的形式。十九世紀的數學家相繼投入，切比雪夫(Chebyshev)首先證明如果 π(x)的極限存在，那麼 A=1 且 B=0，但是沒有成功解決這個猜想；黎曼也研究過這個問題，無奈英年早逝。

一百年後的 1896 年 Hadamard 和 de la Vallée Poussin 以相似的方法分別給出了第一個證明(他們主要利用 ζ(z)在 Re z ≥ 1 時沒有零點的性質與使用 Dirichlet series 來估計 ζ 在無窮遠處的行為[1])，從此質數猜想變成了質數定理。1949 年 Erdos 給出了僅使用組合數學與數論的初等證明[2]。

1980 年 D. J. Newman 給出了目前為止最簡潔的證法，只需要一些基本性質與線積分上的技巧。

二、Notation

- $z = \mathrm{Re}(z) + i^* \mathrm{Im}(z)$，Re for real part、Im for imaginary part。
- $f \sim g$ if $\lim_{x\to\infty} \frac{f}{g} = 1$
- $\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$ Rammen Zeta Function。
- $\Phi(s) := \sum_{p}^{\infty} \frac{\log p}{p^s}$ p is prime, take summation along all prime number。
- $\theta(x) := \sum_{p \le x} \log p$
- O(·): f = O(g) if there is a constant c s.t. for all x : f(x) ≤ c*g(x)

三、Basic property

(0) ζ(s) and Φ(s) absolutely and locally uniformly convergent for Re s >1.

$$\sum_{n=1}^{\infty} \frac{1}{n^{Re(s)}} \ge \sum_{n=1}^{\infty} \left|\frac{1}{n^s}\right| \ge |\zeta(s)| > 0 \text{ also } \sum_{n=1}^{\infty} \left|\frac{1}{n^s}\right| \ge |\Phi(s)| > 0$$

By (0) we have (1) holds for Re s >1.

(1) $\zeta(s) = (1 + \frac{1}{2^s} + \frac{1}{2^{2s}} \ldots)(1 + \frac{1}{3^s} + \frac{1}{3^{2s}} \ldots) \ldots = \prod_p \sum_{k=0}^{\infty} \frac{1}{p^{ks}} = \prod_p \frac{1}{1-p^{-s}}$

(2) $\zeta(s) - \frac{1}{s-1}$ extends holomorphically to the region Re s > 0 :

$$\zeta(s) - \frac{1}{s-1} = \sum_{n=1}^{\infty}\frac{1}{n^s} - \int_1^{\infty}\frac{dx}{x^s} = \sum_{n=1}^{\infty}\int_n^{n+1}\left(\frac{1}{n^s}-\frac{1}{x^s}\right)dx$$

which converges absolutely for Re s > 0

$$\left|\sum_{n=1}^{\infty}\int_n^{n+1}\left(\frac{1}{n^s}-\frac{1}{x^s}\right)dx\right| = \left|s\int_n^{n+1}\int_n^{x}\frac{du}{u^{s+1}}dx\right| \le \max_{n\le y\le n+1}\left|\frac{s}{u^{s+1}}\right| = \frac{|s|}{u^{Re(s)+1}}$$

(3) θ(x) = O(x)

- $\theta(x) = \sum_{p\le x} log\ \ p = \log\prod_{p\le x} p$
- $2^{2n} = \binom{2n}{0} + \binom{2n}{1} + \cdots \binom{2n}{2n} \ge \binom{2n}{n} \ge \prod_{n\le p\le 2n} p = e^{\theta(2n)-\theta(n)}$

It's an easy fact that $\binom{2n}{n}$ is an integer, so n! divides $\frac{2n}{n!}$.

Meanwhile $\prod_{n\le p\le 2n} p$ divides $\frac{2n}{n!}$ and n! not divides $\prod_{n\le p\le 2n} p$

So $\prod_{n\le p\le 2n} p$ divides $\binom{2n}{n}$, hence $\binom{2n}{n} \ge \prod_{n\le p\le 2n} p = e^{\theta(2n)-\theta(n)}$

- $\theta(x)=\sum_{p\le x}\log p$
  $\Rightarrow\ \theta(x)+\log(x+1) \ge \sum_{p\le x+1}\log p = \theta(x+1)$
- $\log(2^{2n}) \ge \log(e^{\theta(2n)-\theta(n)})$
  $\Rightarrow 2n\log 2 \ \ge\ \theta(2n)-\theta(n)$

$\Rightarrow\ \theta(x)\text{-}O(1)\ \approx\ \theta(x)\text{-}\ \theta(x//2)+\theta(x//2)\text{-}\ \theta(x//4)\ldots.\ O(1)$
$\le\ (x\log 2 + x\log 2//2+ x\log 2//4\ldots.\ 2\log 2)$
$\le\ 2x\log 2$

(4) $\zeta(s)\neq 0$ and $\Phi(s)-\frac{1}{s-1}$ is holomorphic for $Re(s)\ge 1$

For Re(s)>1, due to the convergent of product $\prod_p \frac{1}{1-p^{-s}}$ in (1), we have

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_p \frac{\log p}{p^s-1} = \Phi(s) + \sum_p \frac{\log p}{p^s(p^s-1)}$$

- $\sum_p \frac{\log p}{p^s(p^s-1)}$ converges for $Re(s) > \frac{1}{2}$ which means Φ extends

  meromorphically to $Re(s) > \frac{1}{2}$ by (2).

- $\sum_p \frac{\log p}{p^s-1}$ does not converge while s = 1 .

$\cdot \frac{\zeta'(s)}{\zeta(s)}$ has poles for every zero of $\zeta(s)$

Suppose $\zeta(s)$ has a zero of order u at $s = 1 \pm i\alpha$ and a zero of order v at $s = 1 \pm 2i\alpha$ (α is real and $\alpha>0$. u, v ≥ 0 by (2)).

By applying Cauchy's integral formula on $-\frac{\zeta'(s)}{\zeta(s)}$ we have :

$$lim_{\varepsilon\to 0}\ \ \varepsilon\Phi(1+\varepsilon)=1$$
$$lim_{\varepsilon\to 0}\ \ \varepsilon\Phi(1+\varepsilon \pm i\alpha) = -u$$
$$lim_{\varepsilon\to 0}\ \ \varepsilon\Phi(1+\varepsilon \pm 2i\alpha) = -v$$

By definition of Φ :

$$\sum_{r=-2}^{2}\binom{4}{2+r}\Phi(1+\varepsilon \pm r * i\alpha) = \sum_{p}\frac{log\, p}{p^{1+\varepsilon}}(p^{\frac{i\alpha}{2}} + p^{-i\alpha/2})^4 \geq 0$$

And then applying limit of ε on first term above :

$$\binom{4}{0} * (-v) + \binom{4}{1} * (-u) + \binom{4}{2} * 1 + \binom{4}{3} * (-u) + \binom{4}{4} * (-v) \geq 0$$
$$=> \ \ 6 - 8u - 2v \geq 0$$

As definition, u, v ≥ 0, we can conclude u = 0, i.e. $\zeta(1 \pm i\alpha) \neq 0$.

By showing ζ has no zero whose real part is 1, we can conclude that the only pole of Φ satisfied Re(s)=1 is s = 1 . Since

$$lim_{\varepsilon\to 0}\ \ \varepsilon\Phi(1+\varepsilon) = 1$$

This pole can be removed and we have $\Phi(s) - \frac{1}{s-1}$ is holomorphic while Re(s)=1.

Lemma:

Let f(t) ($t \geq 0$) be a bounded and locally integrable function and suppose that the function $g(z) = \int_0^\infty f(t)e^{-zt}dt$ extends holomorphically from $Re(z) > 0$ to $Re(z) \geq 0$. Then $\int_0^\infty f(t)dt$ exists and equals to g(0)

(5) $\int_1^\infty \frac{\theta(x)-x}{x^2}dx$ is a convergent integral .

For Re(s) >1, we have

$$\Phi(s) = \sum_{p}\frac{log\ \ p}{p^s} = \int_1^\infty \frac{d\theta(x)}{x^s} = s\int_1^\infty \frac{\theta(x)}{x^{s+1}}dx = s\int_0^\infty e^{-st}\theta(e^t)dt$$

Take f(t)= $\theta(e^t)e^{-t}-1$ and g(z) = $\frac{\Phi(z+1)}{z+1}-\frac{1}{z}$. (3) implies f bounded.

(4) implies g expands holomorphically to $\mathrm{Re}(z) \geq 0$.

According to Lemma :

$$g(0)=\int_0^\infty (\theta(e^t)e^{-t}-1)dt=\int_0^\infty (\theta(e^t)e^{-2t}-e^{-t})de^t$$

$$=\int_1^\infty (\theta(x)x^{-2}-x^{-1})dx$$

四、proof

Assume that for some λ>1 there are arbitrarily large x s.t. $\theta(x) \geq \lambda x$.

θ is non-decreasing , we have :

$$\int_x^{\lambda x} \frac{\theta(x)-x}{x^2}dx \geq \int_x^{\lambda x} \frac{\lambda x-x}{x^2}dx=\int_1^{\lambda} \frac{\lambda-t}{t^2}dt>0$$

Notice that $\int_1^\lambda \frac{\lambda-t}{t^2}dt$ is irrelevant to x and hence it contradicts to the convergence in (5). Similarly, inequality $\theta(x) \leq \lambda x$ would imply :

$$\int_{\lambda x}^{x} \frac{\theta(x)-x}{x^2}dx \leq \int_{\lambda x}^{x} \frac{\lambda x-x}{x^2}dx=\int_{\lambda}^{1} \frac{\lambda-t}{t^2}dt<0$$

Again a contradiction for λ fixed and x big enouph.

The prime number theorem follows easily from (6), since for any $\varepsilon>0$:

$$\theta(x)=\sum_{p \leq x} log\, p \leq \sum_{p \leq x} log\, x=\pi(x)\log x$$

$$\theta(x) \geq \sum_{x^{1-\varepsilon} \leq p \leq x} log\, p \geq \sum_{x^{1-\varepsilon} \leq p \leq x} (1-\varepsilon) * log\, x=(1-\varepsilon)\log x\,[\pi(x)+O(x^{1-\varepsilon})]$$

then we get

$$\theta(x) \sim \pi(x)\log x$$

and finally

$$\pi(x) \sim \frac{\theta}{\log x} \sim \frac{x}{\log x}$$

五、Lemma:

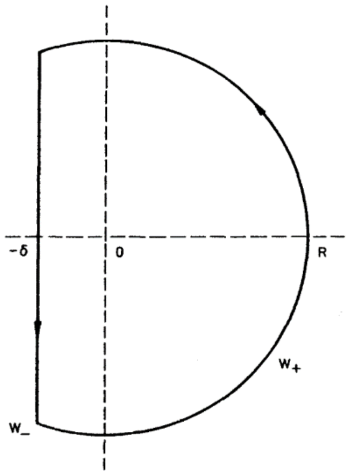
Let f(t) $(t \geq 0)$ be a bounded and locally integrable function and suppose that the function $g(z) = \int_0^\infty f(t)e^{-zt}dt$ extends holomorphically from $Re(z) > 0$ to $Re(z) \geq 0$.

Then $\int_0^\infty f(t)dt$ exists and equals to g(0)

Proof:

Set $g_T(z) = \int_0^T f(t)e^{-zt}dt$, this is clearly holomorphic for all z. we must show that $\lim_{T\to\infty} g_T(0)$=g(z)

Let R be large and let D be the boundary of the region $\{z \in C \mid |z| \leq R, Re(z) \geq -\delta\}$ where $\delta > 0$ is small enough (depending on R) so that g(z) is holomorphic in and on D.



Recall Cauchy's theorem, g and g_T are both holomorphic :

$$g(0) - g_T(0) = 2\pi i \oint_C \frac{g(z) - g_T(z)}{z} dt$$

There is an ingenious trick that multiply $e^{Tz}(1+\frac{z^2}{R^2})$ to $g(z) - g_T(z)$, notice that $e^{Tz}(1+\frac{z^2}{R^2})$ is holomorphic over the whole C.

$$2\pi i \oint_C \frac{(g(z) - g_T(z))e^{Tz}(1+\frac{z^2}{R^2})}{z} dt = \left(g(0) - g_T(0)\right)e^{T*0}\left(1+\frac{0^2}{R^2}\right)$$

And it's still $g(0) - g_T(0)$.

On the semicircle $D_+ = D \cap \{Re(z) > 0\}$ the integrand is bounded by $2B/R^2$, where $B = \max_{t\geq 0} f(t)$ because :

$$|g(z) - g_T(z)| = \left|\int_T^\infty f(t)e^{-zt}dt\right| \leq B\int_T^\infty |e^{-zt}|dt = \frac{B * e^{-Re(z)T}}{Re(z)} \quad (Re(z) > 0)$$

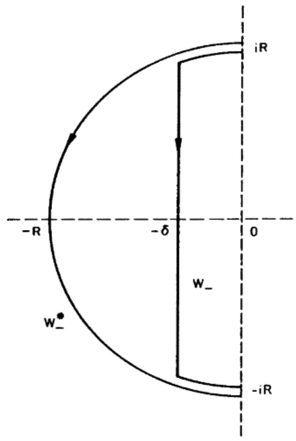
And

$$\left|\frac{e^{zT}(1+\frac{z^2}{R^2})}{z}\right| = \frac{e^{Re(z)T} * 2Re(z)}{R^2} \quad \left(\frac{1+\frac{z^2}{R^2}}{z} = \frac{\left(\frac{R}{z}+\frac{z}{R}\right)}{R} = \frac{2Re\left(\frac{z}{R}\right)}{R} = \frac{2Re(z)}{R^2} \right)$$

$$\frac{2B}{R^2} = \frac{B * e^{-Re(z)T}}{Re(z)} * \frac{2Re(z)}{R^2}$$

Thus the contribution to $g(0)-g_T(0)$ from the integral over D_+ is bounded in absolute value by $B/R > 2B/R^2$ while R is large.

For $D_- = D \cap \{Re(z) < 0\}$, we look at g and g_T separately. Since g_T is entire, the path of integration for the integral involving g_T can be replaced by the semicircle $D'_- = \{Re(z) < 0$ and $|z|<R\}$ and it's integral over D'_- is then bounded in absolute value $\frac{2\pi B}{R}$ by exact same estimate as before.



Finally, the remaining integral over D_- tends to be 0 as T→∞ because the integrand is the product of the function $\frac{g(z)(1+\frac{z^2}{R^2})}{z}$ which is independent of T and e^{zT} which goes to 0 rapidly and uniformly on compact sets as $T \to \infty$ in the half plane Re(z)<0 hence $\lim_{T\to\infty} sup|g(0) - g_T(z)| \leq 2B/R$.

Since R is arbitrary this proves the Lemma.

六、Summary

D. J. Newman 用巧妙的技巧以高斯積分和平地證明了引理，在他之前的證明必須小心地處理 zeta function 或者其 fourier transform 在無窮遠處的行為。

$\pi \sim \frac{x}{\log(x)}$ 收斂的速度算蠻慢的，它在10^5時的誤差大概 10%，它在10^5時的誤差仍有 2%；當然還有更強的版本，不過必須和黎曼猜想掛勾。

質數定理是一個好而簡明的例子，展示ζ(s)如何在複分析與分析之間搭起一座橋梁，也展示其重要性。

七、Reference

· D. Zagier, Newman's Short Proof of the Prime Number Theorem, *Amer. Math. Monthly*, 104(1997), 705-708.

· J. Korevaar, On Newman’s quick way to the prime number theorem, *Math. Intelligencer* 4, 3(1982), 108-115