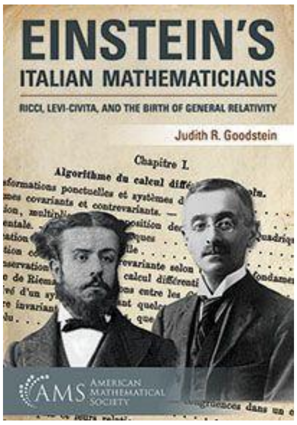
§ Tullio Levi-Civita 1873-1941

Gregorio Ricci Curbastro 1853-1925

1915 年三月到五月，Levi-Civita 與愛因斯坦書信連絡，對愛因斯坦的”Entwurf”理論(for general theory of relativity)表示反對。

After mentioning the fundamental geometric results of Beltrami and Ricci-Curbastro，the manuscript of Lampariello specifies the decisive – mathematically and physically – contributions by Levi-Civita to general relativity。

1915 年秋天，David Hilbert 發現愛因斯坦 1914 年推導的缺陷，1916 年 3 月 30 日愛因斯坦回信表示接受：「The error you found in my paper of 1914 has now become completely clear to me。」



§ Levi-Civita connection ∇

1. $Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$ ∇與 g 相容

2. $\nabla_X Y - \nabla_Y X = [X,Y]$ torsion free

Levi-Civita 定理

(M,g) 是 Riemann manifold 則∃!一個對稱且與 g 相容的 connection，稱為 Levi-Civita connection。

§ parallel transport

Example

1. 球面上的平行移動與傅科擺

Leon Foucault(1819~1868) Gregorio Ricci(1853~1925) Levi-Civita(1873~1941)

Notes

Christoffel symbol $\Gamma^i_{jk} = \frac{1}{2} g^{il} (\frac{\partial g_{kl}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l})$

Connection $\nabla_X Y = \sum_i (XY^i + \sum_{jk} \Gamma^i_{jk} X^j Y^k)$

Curvature $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$

$$R^{\rho}_{\sigma\mu\nu} = \partial_{\mu}\Gamma^{\rho}_{\nu\sigma} - \partial_{\nu}\Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\sigma\nu} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}$$

Bianchi identities

Einstein equation $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$

Geodesic equation $\frac{d^2x^{\mu}}{d\lambda^2} + \Gamma^{\mu}_{\rho\sigma}\frac{dx^{\rho}}{d\lambda}\frac{dx^{\sigma}}{d\lambda} = 0$

Energy-momentum tensor

Covariant divergence

Exercises

1. (R^2, g)的極座標 $g = ds^2 = dr^2 + r^2d\theta^2$，求$\Gamma^k_{ij} =$

   $(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}, (g^{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & r^{-2} \end{pmatrix}$

   $\Gamma^1_{22} = -r, \Gamma^2_{12} = \frac{1}{r}$，其它皆為 0

2. S^2 $X(\theta,\varphi) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$

   (1) Prove that $g = d\theta^2 + \sin^2\theta d\varphi^2$

   (2) Find $\Gamma^i_{jk} =$

   (3) Find Ricci tensor and Ricci scalar R

   $\Gamma^1_{22} = -\sin\theta\cos\theta, \Gamma^2_{12} = \Gamma^2_{21} = \cot\theta$ $R_{11} = 1, R_{12} = R_{21} = 0, R_{22} = \sin^2\theta$

   $R = \frac{2}{a^2}$ (for $ds^2 = a^2(d\theta^2 + \sin^2\theta d\varphi^2)$)

3. A Levi-Civita connection preserves length and angles under parallel transport。
   Let $T = \alpha'(t)$ be tangent to curve $\alpha(t)$

   X,Y be parallel transport along α，i.e. $\nabla_T X = \nabla_T Y = 0$

   Then $\nabla_T < X, X > = < \nabla_T, X > + < X, \nabla_T X > = 0$，so $\|X\|$ is a constant。

   $\nabla_T < X, Y > = < \nabla_T X, Y > + < X, \nabla_T Y > = 0$，<X.Y> is a constant

$$\therefore \cos\theta = \frac{<X,Y>}{\|X\|\|Y\|} = constant$$ 。

4.

後記：

1. 用 Euler-Lagragain equation 算出 geodesic，由此找到 Christoffel symbols。
2. A geodesic is a curve along which the tangent vector is parallel transport。
3. Foucault 與 Levi-Civita 張海潮
4. Correspondence between Einstein and Levi-Civita by Galina Weinstein
5. Einstein's Italian Mathematicians：Ricci，Levi-Civita and the Birth of General Relativity by Judith R. Goodstein