Goldstein Classical Mechanics Notes

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1 Chapter 1: Elementary Principles

1.1 Mechanics of a Single Particle

Classical mechanics incorporates special relativity. 'Classical' refers to the contradistinction to 'quantum' mechanics.

Velocity:

$$\mathbf{v} = \frac{d\mathbf{r}}{dt}.$$

Linear momentum:

$$\mathbf{p} = m\mathbf{v}.$$

Force:

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}.$$

In most cases, mass is constant and force is simplified:

$$\mathbf{F} = \frac{d}{dt}(m\mathbf{v}) = m\frac{d\mathbf{v}}{dt} = m\mathbf{a}.$$

Acceleration:

$$\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2}.$$

Newton's second law of motion holds in a reference frame that is inertial or Galilean.

Angular Momentum:

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}.$$

Torque:

$$\mathbf{T} = \mathbf{r} \times \mathbf{F}.$$

Torque is the time derivative of angular momentum:

$$\mathbf{T} = \frac{d\mathbf{L}}{dt}.$$

Work:

$$W_{12} = \int_1^2 \mathbf{F} \cdot d\mathbf{r}.$$

In most cases, mass is constant and work simplifies to:

$$W_{12} = m\int_1^2 \frac{d\mathbf{v}}{dt} \cdot \mathbf{v}dt = m\int_1^2 \mathbf{v} \cdot \frac{d\mathbf{v}}{dt}dt = m\int_1^2 \mathbf{v} \cdot d\mathbf{v}$$

$$W_{12} = \frac{m}{2}(v_2^2 - v_1^2) = T_2 - T_1$$

Kinetic Energy:

$$T = \frac{mv^2}{2}$$

The work is the change in kinetic energy.

A force is considered conservative if the work is the same for any physically possible path. Independence of W_{12} on the particular path implies that the work done around a closed ciruit is zero:

$$\oint \mathbf{F} \cdot d\mathbf{r} = 0$$

If friction is present, a system is non-conservative.

Potential Energy:

$$\mathbf{F} = -\nabla V(\mathbf{r}).$$

The capacity to do work that a body or system has by viture of is position is called its potential energy. V above is the potential energy. To express work in a way that is independent of the path taken, a change in a quantity that depends on only the end points is needed. This quantity is potential energy. Work is now $V_1 - V_2$. The change is -V.

Energy Conservation Theorem for a Particle: If forces acting on a particle are conservative, then the total energy of the particle, T + V, is conserved.

The Conservation Theorem for the Linear Momentum of a Particle states that linear momentum, **p**, is conserved if the total force **F**, is zero.

The Conservation Theorem for the Angular Momentum of a Particle states that angular momentum, **L**, is conserved if the total torque **T**, is zero.

1.2 Mechanics of Many Particles

Newton's third law of motion, equal and opposite forces, does not hold for all forces. It is called the weak law of action and reaction.

Center of mass:

$$\mathbf{R} = \frac{\sum m_i \mathbf{r}_i}{\sum m_i} = \frac{\sum m_i \mathbf{r}_i}{M}.$$

Center of mass moves as if the total external force were acting on the entire mass of the system concentrated at the center of mass. Internal forces that obey Newton's third law, have no effect on the motion of the center of mass.

$$\mathbf{F}^{(e)} \equiv M\frac{d^2\mathbf{R}}{dt^2} = \sum_i \mathbf{F}_i^{(e)}.$$

Motion of center of mass is unaffected. This is how rockets work in space.

Total linear momentum:

$$\mathbf{P} = \sum_i m_i \frac{d\mathbf{r}_i}{dt} = M\frac{d\mathbf{R}}{dt}.$$

Conservation Theorem for the Linear Momentum of a System of Particles: If the total external force is zero, the total linear momentum is conserved.

The strong law of action and reaction is the condition that the internal forces between two particles, in addition to being equal and opposite, also lie along the line joining the particles. Then the time derivative of angular momentum is the total external torque:

$$\frac{d\mathbf{L}}{dt} = \mathbf{N}^{(e)}.$$

Torque is also called the moment of the external force about the given point.

Conservation Theorem for Total Angular Momentum: $\mathbf{L}$ is constant in time if the applied torque is zero.

Linear Momentum Conservation requires weak law of action and reaction.

Angular Momentum Conservation requires strong law of action and reaction.

Total Angular Momentum:

$$\mathbf{L} = \sum_i \mathbf{r}_i \times \mathbf{p}_i = \mathbf{R} \times M\mathbf{v} + \sum_i \mathbf{r}'_i \times \mathbf{p}'_i.$$

Total angular momentum about a point O is the angular momentum of motion concentrated at the center of mass, plus the angular momentum of motion about the center of mass. If the center of mass is at rest wrt the origin then the angular momentum is independent of the point of reference.

Total Work:

$$W_{12} = T_2 - T_1$$

where T is the total kinetic energy of the system: $T = \frac{1}{2}\sum_i m_i v_i^2$.
Total kinetic energy:

$$T = \frac{1}{2}\sum_i m_i v_i^2 = \frac{1}{2}Mv^2 + \frac{1}{2}\sum_i m_i v_i'^2.$$

Kinetic energy, like angular momentum, has two parts: the K.E. obtained if all the mass were concentrated at the center of mass, plus the K.E. of motion about the center of mass.

Total potential energy:

$$V = \sum_i V_i + \frac{1}{2}\sum_{i,j \; i\neq j} V_{ij}.$$

If the external and internal forces are both derivable from potentials it is possible to define a total potential energy such that the total energy $T + V$ is conserved.

The term on the right is called the internal potential energy. For rigid bodies the internal potential energy will be constant. For a rigid body the internal forces do no work and the internal potential energy remains constant.

1.3 Constraints

- *holonomic constraints*: think rigid body, think $f(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, ..., t) = 0$, think a particle constrained to move along any curve or on a given surface.
- *nonholonomic constraints*: think walls of a gas container, think particle placed on surface of a sphere because it will eventually slide down part of the way but will fall off, not moving along the curve of the sphere.

1. *rheonomous constraints*: time is an explicit variable...example: bead on moving wire
2. *scleronomous constraints*: equations of contraint are NOT explicitly dependent on time...example: bead on rigid curved wire fixed in space

Difficulties with constraints:

1. Equations of motion are not all independent, because coordinates are no longer all independent

2. Forces are not known beforehand, and must be obtained from solution.

For *holonomic constraints* introduce **generalized coordinates**. Degrees of freedom are reduced. Use independent variables, eliminate dependent coordinates. This is called a *transformation*, going from one set of dependent variables to another set of independent variables. Generalized coordinates are worthwhile in problems even without constraints.

Examples of generalized coordinates:

1. Two angles expressing position on the sphere that a particle is constrained to move on.

2. Two angles for a double pendulum moving in a plane.

3. Amplitudes in a Fourier expansion of $\mathbf{r}_j$.

4. Quanities with with dimensions of energy or angular momentum.

For *nonholonomic constraints* equations expressing the constraint cannot be used to eliminate the dependent coordinates. Nonholonomic constraints are HARDER TO SOLVE.

1.4 D'Alembert's Principle and Lagrange's Equations

Developed by D'Alembert, and thought of first by Bernoulli, the principle that:

$$\sum_i (\mathbf{F}_i^{(a)} - \frac{d\mathbf{p}_i}{dt}) \cdot \delta \mathbf{r}_i = 0$$

This is valid for systems which virtual work of the forces of constraint vanishes, like rigid body systems, and no friction systems. This is the only restriction on the nature of the constraints: workless in a virtual displacement. This is again D'Alembert's principle for the motion of a system, and what is good about it is that the forces of constraint are not there. This is great news, but it is not yet in a form that is useful for deriving equations of motion. Transform this equation into an expression involving virtual displacements of the generalized coordinates. The generalized coordinates are independent of each other for holonomic constraints. Once we have the expression in terms of generalized coordinates the coefficients of the δq_i can be set separately equal to zero. The result is:

$$\sum \{ [\frac{d}{dt}(\frac{\partial T}{\partial \dot{q}_j}) - \frac{\partial T}{\partial q_j}] - Q_j \} \delta q_j = 0$$

Lagrange's Equations come from this principle. If you remember the individual coefficients vanish, and allow the forces derivable from a scaler potential function, and forgive me for skipping some steps, the result is:

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_j}) - \frac{\partial L}{\partial q_j} = 0$$

1.5 Velocity-Dependent Potentials and The Dissipation Function

The velocity dependent potential is important for the electromagnetic forces on moving charges, the electromagnetic field.

$$L = T - U$$

where U is the generalized potential or velocity-dependent potential.

For a charge mvoing in an electric and magnetic field, the Lorentz force dictates:

$$\mathbf{F} = q[\mathbf{E} + (\mathbf{v} \times \mathbf{B})].$$

The equation of motion can be dervied for the x-dirction, and notice they are identical component wise:

$$m\ddot{x} = q[E_x + (\mathbf{v} \times \mathbf{B})_x].$$

If frictional forces are present(not all the forces acting on the system are derivable from a potential), Lagrange's equations can always be written:

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_j}) - \frac{\partial L}{\partial q_j} = Q_j.$$

where Q_j represents the forces not arising from a potential, and L contains the potential of the conservative forces as before.

Friction is commonly,

$$F_{fx} = -k_x v_x.$$

Rayleigh's dissipation function:

$$F_{dis} = \frac{1}{2}\sum_i (k_x v_{ix}^2 + k_y v_{iy}^2 + k_z v_{iz}^2).$$

The total frictional force is:

$$\mathbf{F}_f = -\nabla_v F_{dis}$$

Work done by system against friction:

$$dW_f = -2F_{dis}dt$$

The rate of energy dissipation due to friction is $2F_{dis}$ and the component of the generalized force resulting from the force of friction is:

$$Q_j = -\frac{\partial F_{dis}}{\partial \dot{q}_j}.$$

In use, both L and F_{dis} must be specified to obtain the equations of motion:

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_j}) - \frac{\partial L}{\partial q_j} = -\frac{\partial F_{dis}}{\partial \dot{q}_j}.$$

1.6 Applications of the Lagrangian Formulation

The Lagrangian method allows us to eliminate the forces of constraint from the equations of motion. Scalar functions T and V are much easier to deal with instead of vector forces and accelerations.

Procedure:

1. Write T and V in generalized coordinates.
2. Form L from them.
3. Put L into Lagrange's Equations
4. Solve for the equations of motion.

Simple examples are:

1. a single particle is space(Cartesian coordinates, Plane polar coordinates)
2. atwood's machine
3. a bead sliding on a rotating wire(time-dependent constraint).

Forces of contstraint, do not appear in the Lagrangian formulation. They also cannot be directly derived.

Goldstein Chapter 1 Derivations

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1 Derivations

1. Show that for a single particle with constant mass the equation of motion implies the follwing differential equation for the kinetic energy:

$$\frac{dT}{dt} = \mathbf{F} \cdot \mathbf{v}$$

while if the mass varies with time the corresponding equation is

$$\frac{d(mT)}{dt} = \mathbf{F} \cdot \mathbf{p}.$$

Answer:

$$\frac{dT}{dt} = \frac{d(\frac{1}{2}mv^2)}{dt} = m\mathbf{v} \cdot \dot{\mathbf{v}} = m\mathbf{a} \cdot \mathbf{v} = \mathbf{F} \cdot \mathbf{v}$$

with time variable mass,

$$\frac{d(mT)}{dt} = \frac{d}{dt}(\frac{p^2}{2}) = \mathbf{p} \cdot \dot{\mathbf{p}} = \mathbf{F} \cdot \mathbf{p}.$$

2. Prove that the magnitude R of the position vector for the center of mass from an arbitrary origin is given by the equation:

$$M^2R^2 = M\sum_i m_i r_i^2 - \frac{1}{2}\sum_{i,j} m_i m_j r_{ij}^2.$$

Answer:

$$M\mathbf{R} = \sum m_i \mathbf{r}_i$$

$$M^2\mathbf{R}^2 = \sum_{i,j} m_i m_j \mathbf{r}_i \cdot \mathbf{r}_j$$

Solving for $\mathbf{r}_i \cdot \mathbf{r}_j$ realize that $\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j$. Square $\mathbf{r}_i - \mathbf{r}_j$ and you get

$$r_{ij}^2 = r_i^2 - 2\mathbf{r}_i \cdot \mathbf{r}_j + r_j^2$$

Plug in for $\mathbf{r}_i \cdot \mathbf{r}_j$

$$\mathbf{r}_i \cdot \mathbf{r}_j = \frac{1}{2}(r_i^2 + r_j^2 - r_{ij}^2)$$

$$M^2R^2 = \frac{1}{2}\sum_{i,j} m_i m_j r_i^2 + \frac{1}{2}\sum_{i,j} m_i m_j r_j^2 - \frac{1}{2}\sum_{i,j} m_i m_j r_{ij}^2$$

$$M^2R^2 = \frac{1}{2}M\sum_{i} m_i r_i^2 + \frac{1}{2}M\sum_{j} m_j r_j^2 - \frac{1}{2}\sum_{i,j} m_i m_j r_{ij}^2$$

$$M^2R^2 = M\sum_{i} m_i r_i^2 - \frac{1}{2}\sum_{i,j} m_i m_j r_{ij}^2$$

3. Suppose a system of two particles is known to obey the equations of motions,

$$M\frac{d^2\mathbf{R}}{dt^2} = \sum_i \mathbf{F}_i^{(e)} \equiv \mathbf{F}^{(e)}$$

$$\frac{d\mathbf{L}}{dt} = \mathbf{N}^{(e)}$$

From the equations of the motion of the individual particles show that the internal forces between particles satisfy both the weak and the strong laws of action and reaction. The argument may be generalized to a system with arbitrary number of particles, thus proving the converse of the arguments leading to the equations above.

Answer:

First, if the particles satisfy the strong law of action and reaction then they will automatically satisfy the weak law. The weak law demands that only the forces be equal and opposite. The strong law demands they be equal and opposite and lie along the line joining the particles. The first equation of motion tells us that internal forces have no effect. The equations governing the individual particles are

$$\dot{\mathbf{p}}_1 = \mathbf{F}_1^{(e)} + \mathbf{F}_{21}$$

$$\dot{\mathbf{p}}_2 = \mathbf{F}_2^{(e)} + \mathbf{F}_{12}$$

Assuming the equation of motion to be true, then

$$\dot{\mathbf{p}}_1 + \dot{\mathbf{p}}_2 = \mathbf{F}_1^{(e)} + \mathbf{F}_{21} + \mathbf{F}_2^{(e)} + \mathbf{F}_{12}$$

must give

$$\mathbf{F}_{12} + \mathbf{F}_{21} = 0$$

Thus $F_{12} = -F_{21}$ and they are equal and opposite and satisfy the weak law of action and reaction. If the particles obey

$$\frac{d\mathbf{L}}{dt} = \mathbf{N}^{(e)}$$

then the time rate of change of the total angular momentum is only equal to the total external torque; that is, the internal torque contribution is null. For two particles, the internal torque contribution is

$$\mathbf{r}_1 \times \mathbf{F}_{21} + \mathbf{r}_2 \times \mathbf{F}_{12} = \mathbf{r}_1 \times \mathbf{F}_{21} + \mathbf{r}_2 \times (-\mathbf{F}_{21}) = (\mathbf{r}_1 - \mathbf{r}_2) \times \mathbf{F}_{21} = \mathbf{r}_{12} \times \mathbf{F}_{21} = 0$$

Now the only way for $\mathbf{r}_{12} \times \mathbf{F}_{21}$ to equal zero is for both $\mathbf{r}_{12}$ and $\mathbf{F}_{21}$ to lie on the line joining the two particles, so that the angle between them is zero, ie the magnitude of their cross product is zero.

$$\mathbf{A} \times \mathbf{B} = ABsin\theta$$

4. The equations of constraint for the rolling disk,

$$dx - a \sin\theta d\psi = 0$$

$$dy + a \cos\theta d\psi = 0$$

are special cases of general linear differential equations of constraint of the form

$$\sum_{i=1}^{n} g_i(x_1, \ldots, x_n) dx_i = 0.$$

A constraint condition of this type is holonomic only if an integrating function $f(x_1, \ldots, x_n)$ can be found that turns it into an exact differential. Clearly the function must be such that

$$\frac{\partial(fg_i)}{\partial x_j} = \frac{\partial(fg_j)}{\partial x_i}$$

for all $i \neq j$. Show that no such integrating factor can be found for either of the equations of constraint for the rolling disk.

Answer:

First attempt to find the integrating factor for the first equation. Note it is in the form:

$$Pdx + Qd\phi + Wd\theta = 0$$

where P is 1, Q is $-a\sin\theta$ and W is 0. The equations that are equivalent to

$$\frac{\partial(fg_i)}{\partial x_j} = \frac{\partial(fg_j)}{\partial x_i}$$

are

$$\frac{\partial(fP)}{\partial\phi} = \frac{\partial(fQ)}{\partial x}$$
$$\frac{\partial(fP)}{\partial\theta} = \frac{\partial(fW)}{\partial x}$$
$$\frac{\partial(fQ)}{\partial\theta} = \frac{\partial(fW)}{\partial\phi}$$

These are explicitly:

$$\frac{\partial(f)}{\partial\phi} = \frac{\partial(-fa\sin\theta)}{\partial x}$$
$$\frac{\partial(f)}{\partial\theta} = 0$$
$$\frac{\partial(-fa\sin\theta)}{\partial\theta} = 0$$

Simplfying the last two equations yields:

$$f\cos\theta = 0$$

Since y is not even in this first equation, the integrating factor does not depend on y and because of $\frac{\partial f}{\partial\theta} = 0$ it does not depend on θ either. Thus

$$f = f(x, \phi)$$

The only way for f to satisfy this equation is if f is constant and thus apparently there is no integrating function to make these equations exact. Performing the same procedure on the second equation you can find

$$\frac{\partial(fa\cos\theta)}{\partial y} = \frac{\partial f}{\partial\phi}$$

$$a\cos\theta\frac{\partial f}{\partial y} = \frac{\partial f}{\partial\phi}$$

and

$$f\sin\theta = 0$$

$$\frac{\partial f}{\partial \theta} = 0$$

leading to

$$f = f(y, \phi)$$

and making it impossible for f to satsify the equations unless as a constant. If this question was confusing to you, it was confusing to me too. Mary Boas says it is 'not usually worth while to spend much time searching for an integrating factor' anyways. That makes me feel better.

5. Two wheels of radius a are mounted on the ends of a common axle of length b such that the wheels rotate independently. The whole combination rolls without slipping on a palne. Show that there are two nonholonomic equations of constraint,

$$\cos\theta dx + \sin\theta dy = 0$$

$$\sin\theta dx - \cos\theta dy = \frac{1}{2}a(d\phi + d\phi')$$

(where θ,ϕ, and ϕ' have meanings similar to those in the problem of a single vertical disk, and (x,y) are the corrdinates of a point on the axle midway between the two wheels) and one holonomic equation of constraint,

$$\theta = C - \frac{a}{b}(\phi - \phi')$$

where C is a constant.

Answer:

The trick to this problem is carefully looking at the angles and getting the signs right. I think the fastest way to solve this is to follow the same procedure that was used for the single disk in the book, that is, find the speed of the disk, find the point of contact, and take the derivative of the x component, and y component of position, and solve for the equations of motion. Here the steps are taken a bit further because a holonomic relationship can be found that relates θ, ϕ and ϕ'. Once you have the equations of motion, from there its just slightly tricky algebra. Here goes:

We have two speeds, one for each disk

$$v' = a\dot{\phi}'$$

$$v = a\dot{\phi}$$

and two contact points,

$$(x \pm \frac{b}{2}\cos\theta, y \pm \frac{b}{2}\sin\theta)$$

The contact points come from the length of the axis being b as well as x and y being the center of the axis. The components of the distance are cos and sin for x and y repectively.

So now that we've found the speeds, and the points of contact, we want to take the derivatives of the x and y parts of their contact positions. This will give us the components of the velocity. Make sure you get the angles right, they were tricky for me.

$$\frac{d}{dt}(x + \frac{b}{2}\cos\theta) = v_x$$

$$\dot{x} - \frac{b}{2}\sin\theta\dot{\theta} = v\cos(180 - \theta - 90) = v\cos(90 - \theta) = v\cos(-90 + \theta) = v\sin\theta$$

$$\dot{x} - \frac{b}{2}\sin\theta\dot{\theta} = a\dot{\phi}\sin\theta$$

Do this for the next one, and get:

$$\dot{x} + \frac{b}{2}\sin\theta\dot{\theta} = a\dot{\phi}'\sin\theta$$

The plus sign is there because of the derivative of cos multiplied with the negative for the primed wheel distance from the center of the axis. For the y parts:

$$\frac{d}{dt}(y + \frac{b}{2}\sin\theta) = v_y$$

$$\dot{y} + \frac{b}{2}\cos\theta\dot{\theta} = -v\cos\theta = -a\dot{\phi}\cos\theta$$

It is negative because I decided to have axis in the first quadrent heading south-east. I also have the primed wheel south-west of the non-primed wheel. A picture would help, but I can't do that on latex yet. So just think about it.

Do it for the next one and get:

$$\dot{y} - \frac{b}{2}\cos\theta\dot{\theta} = -a\dot{\phi}'\cos\theta$$

All of the derivatives together so you aren't confused what I just did:

$$\dot{x} - \frac{b}{2}\sin\theta\dot{\theta} = a\dot{\phi}\sin\theta$$

$$\dot{x} + \frac{b}{2}\sin\theta\dot{\theta} = a\dot{\phi}'\sin\theta$$

$$\dot{y} + \frac{b}{2}\cos\theta\dot{\theta} = -a\dot{\phi}\cos\theta$$

$$\dot{y} - \frac{b}{2}\cos\theta\dot{\theta} = -a\dot{\phi}'\cos\theta$$

Now simplify them by cancelling the $dt's$ and leaving the x and y's on one side:

$$dx = \sin\theta[\frac{b}{2}d\theta + ad\phi] \tag{1}$$

$$dx = \sin\theta[-\frac{b}{2}d\theta + ad\phi'] \tag{2}$$

$$dy = -\cos\theta[\frac{b}{2}d\theta + ad\phi] \tag{3}$$

$$dy = -\cos\theta[-\frac{b}{2}d\theta + ad\phi'] \tag{4}$$

Now we are done with the physics. The rest is manipulation of these equations of motion to come up with the constraints. For the holonomic equation use (1)-(2).

$$(1) - (2) = 0 = bd\theta + a(d\phi - d\phi')$$

$$d\theta = -\frac{a}{b}(d\phi - d\phi')$$

$$\theta = -\frac{a}{b}(\phi - \phi') + C$$

For the other two equations, I started with

$$(1)\cos\theta + (3)\sin\theta = \cos\theta\sin\theta[\frac{b}{2}d\theta + ad\phi] - \sin\theta\cos\theta[\frac{b}{2}d\theta + ad\phi]$$

$$\cos\theta dx + \sin\theta dy = 0$$

and

$$(1) + (2) = 2dx = \sin\theta a[d\phi + d\phi']$$

$$(3) + (4) = 2dy = -\cos\theta a[d\phi + d\phi']$$

multiply dy by $-\cos\theta$ and multiply dx by $\sin\theta$ to yield yourself

$$-\cos\theta dy = \cos^2\theta\frac{a}{2}[d\phi + d\phi']$$

$$\sin\theta dx = \sin^2\theta\frac{a}{2}[d\phi + d\phi']$$

Add them together and presto!

$$\sin\theta dx - \cos\theta dy = \frac{a}{2}[d\phi + d\phi']$$

6. A particle moves in the xy plane under the constraint that its velocity vector is always directed towards a point on the x axis whose abscissa is some given function of time $f(t)$. Show that for $f(t)$ differentiable, but otherwise arbitrary, *the constraint is nonholonomic.*

Answer:

The abscissa is the x-axis distance from the origin to the point on the x-axis that the velocity vector is aimed at. It has the distance $f(t)$.

I claim that the ratio of the velocity vector components must be equal to the ratio of the vector components of the vector that connects the particle to the point on the x-axis. The directions are the same. The velocity vector components are:

$$v_y = \frac{dy}{dt}$$

$$v_x = \frac{dx}{dt}$$

The vector components of the vector that connects the particle to the point on the x-axis are:

$$V_y = y(t)$$

$$V_x = x(t) - f(t)$$

For these to be the same, then

$$\frac{v_y}{v_x} = \frac{V_y}{V_x}$$

$$\frac{dy}{dx} = \frac{y(t)}{x(t) - f(t)}$$

$$\frac{dy}{y(t)} = \frac{dx}{x(t) - f(t)}$$

This cannot be integrated with $f(t)$ being arbituary. Thus the constraint is nonholonomic. It's nice to write the constraint in this way because it's frequently the type of setup Goldstein has:

$$ydx + (f(t) - x)dy = 0$$

There can be no integrating factor for this equation.

7. The Lagrangian equations can be written in the form of the Nielsen's equations.

$$\frac{\partial \dot{T}}{\partial \dot{q}} - 2\frac{\partial T}{\partial q} = Q$$

Show this.

Answer:

I'm going to set the two forms equal and see if they match. That will show that they can be written as displayed above.

$$\text{Lagrangian Form} = \text{Nielsen's Form}$$

$$\frac{d}{dt}(\frac{\partial T}{\partial \dot{q}}) - \frac{\partial T}{\partial q} = \frac{\partial \dot{T}}{\partial \dot{q}} - 2\frac{\partial T}{\partial q}$$

$$\frac{d}{dt}(\frac{\partial T}{\partial \dot{q}}) + \frac{\partial T}{\partial q} = \frac{\partial \dot{T}}{\partial \dot{q}} \tag{5}$$

What is $\frac{\partial \dot{T}}{\partial \dot{q}}$ you may ask? Well, lets solve for $\dot{T}$ first.

$$\dot{T} \equiv \frac{d}{dt}T(q, \dot{q}, t)$$

Because $\frac{d}{dt}$ is a full derivative, you must not forget the chain rule.

$$\dot{T} \equiv \frac{d}{dt}T(q, \dot{q}, t) = \frac{\partial T}{\partial t} + \frac{\partial T}{\partial q}\dot{q} + \frac{\partial T}{\partial \dot{q}}\ddot{q}$$

Now lets solve for $\frac{\partial \dot{T}}{\partial \dot{q}}$, not forgetting the product rule

$$\frac{\partial \dot{T}}{\partial \dot{q}} = \frac{\partial}{\partial \dot{q}}[\frac{\partial T}{\partial t} + \frac{\partial T}{\partial q}\dot{q} + \frac{\partial T}{\partial \dot{q}}\ddot{q}]$$

$$\frac{\partial \dot{T}}{\partial \dot{q}} = \frac{\partial}{\partial \dot{q}}\frac{\partial T}{\partial t} + \frac{\partial}{\partial \dot{q}}\frac{\partial T}{\partial q}\dot{q} + \frac{\partial T}{\partial q}\frac{\partial \dot{q}}{\partial \dot{q}} + \frac{\partial}{\partial \dot{q}}\frac{\partial T}{\partial \dot{q}}\ddot{q}$$

$$\frac{\partial \dot{T}}{\partial \dot{q}} = \frac{\partial}{\partial t}\frac{\partial T}{\partial \dot{q}} + \frac{\partial}{\partial q}\frac{\partial T}{\partial \dot{q}}\dot{q} + \frac{\partial T}{\partial q} + \frac{\partial}{\partial \dot{q}}(\frac{\partial T}{\partial \dot{q}})\ddot{q}$$

Now we have $\frac{\partial \dot{T}}{\partial \dot{q}}$, so lets plug this into equation (5).

$$\frac{d}{dt}(\frac{\partial T}{\partial \dot{q}}) + \frac{\partial T}{\partial q} = \frac{\partial}{\partial t}\frac{\partial T}{\partial \dot{q}} + \frac{\partial}{\partial q}\frac{\partial T}{\partial \dot{q}}\dot{q} + \frac{\partial T}{\partial q} + \frac{\partial}{\partial \dot{q}}(\frac{\partial T}{\partial \dot{q}})\ddot{q}$$

$$\frac{d}{dt}(\frac{\partial T}{\partial \dot{q}}) = \frac{\partial}{\partial t}\frac{\partial T}{\partial \dot{q}} + \frac{\partial}{\partial q}\frac{\partial T}{\partial \dot{q}}\dot{q} + \frac{\partial}{\partial \dot{q}}(\frac{\partial T}{\partial \dot{q}})\ddot{q}$$

Notice that this is indeed true.

$$\frac{d}{dt}(\frac{\partial T}{\partial \dot{q}}) = \frac{\partial}{\partial t}(\frac{\partial T}{\partial \dot{q}}) + \frac{\partial}{\partial q}(\frac{\partial T}{\partial \dot{q}})\dot{q} + \frac{\partial}{\partial \dot{q}}(\frac{\partial T}{\partial \dot{q}})\ddot{q}$$

because $T = T(q, \dot{q}, t)$.

If L is a Lagrangian for a system of n degrees of freedom satisfying Lagrange's equations, show by direct substitution that

$$L' = L + \frac{dF(q_1, ..., q_n, t)}{dt}$$

also satisfies Lagrange's equations where F is any arbitrary, but differentiable, function of its arguments.

Answer:

Let's directly substitute L' into Lagrange's equations.

$$\frac{d}{dt}\frac{\partial L'}{\partial \dot{q}} - \frac{\partial L'}{\partial q} = 0$$

$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}}(L + \frac{dF}{dt}) - \frac{\partial}{\partial q}(L + \frac{dF}{dt}) = 0$$

$$\frac{d}{dt}[\frac{\partial L}{\partial \dot{q}} + \frac{\partial}{\partial \dot{q}}\frac{dF}{dt}] - \frac{\partial L}{\partial q} - \frac{\partial}{\partial q}\frac{dF}{dt} = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} + \frac{d}{dt}\frac{\partial}{\partial \dot{q}}\frac{dF}{dt} - \frac{\partial}{\partial q}\frac{dF}{dt} = 0$$

On the left we recognized Lagrange's equations, which we know equal zero. Now to show the terms with F vanish.

$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}}\frac{dF}{dt} - \frac{\partial}{\partial q}\frac{dF}{dt} = 0$$

$$\frac{d}{dt}\frac{\partial \dot{F}}{\partial \dot{q}} = \frac{\partial \dot{F}}{\partial q}$$

This is shown to be true because

$$\frac{\partial \dot{F}}{\partial \dot{q}} = \frac{\partial F}{\partial q}$$

We have

$$\frac{d}{dt}\frac{\partial \dot{F}}{\partial \dot{q}} = \frac{d}{dt}\frac{\partial F}{\partial q}$$

$$= \frac{\partial}{\partial t}\frac{\partial F}{\partial q} + \frac{\partial}{\partial q}\frac{\partial F}{\partial q}\dot{q}$$

$$= \frac{\partial}{\partial q}[\frac{\partial F}{\partial t} + \frac{\partial F}{\partial q}\dot{q}] = \frac{\partial \dot{F}}{\partial q}$$

Thus as Goldstein reminded us, $L = T - V$ is a suitable Lagrangian, but it is not the only Lagrangian for a given system.

9. The electromagnetic field is invariant under a gauge transformation of the scalar and vector potential given by

$$\mathbf{A} \rightarrow \mathbf{A} + \nabla\psi(\mathbf{r}, \mathbf{t})$$

$$\phi \rightarrow \phi - \frac{1}{c}\frac{\partial\psi}{\partial t}$$

where ψ is arbitrary (but differentiable). What effect does this gauge transformation have on the Lagrangian of a particle moving in the electromagnetic field? Is the motion affected?

Answer:

$$L = \frac{1}{2}mv^2 - q\phi + \frac{q}{c}\mathbf{A}\cdot\mathbf{v}$$

Upon the gauge transformation:

$$L' = \frac{1}{2}mv^2 - q[\phi - \frac{1}{c}\frac{\partial\psi}{\partial t}] + \frac{q}{c}[\mathbf{A} + \nabla\psi(r,t)]\cdot\mathbf{v}$$

$$L' = \frac{1}{2}mv^2 - q\phi + \frac{q}{c}\mathbf{A}\cdot\mathbf{v} + \frac{q}{c}\frac{\partial\psi}{\partial t} + \frac{q}{c}\nabla\psi(r,t)\cdot\mathbf{v}$$

$$L' = L + \frac{q}{c}[\frac{\partial\psi}{\partial t} + \nabla\psi(r,t)\cdot\mathbf{v}]$$

$$L' = L + \frac{q}{c}[\dot{\psi}]$$

In the previous problem it was shown that:

$$\frac{d}{dt}\frac{\partial\dot{\psi}}{\partial\dot{q}} = \frac{\partial\dot{\psi}}{\partial q}$$

For ψ differentiable but arbitrary. This is all that you need to show that the Lagrangian is changed but the motion is not. This problem is now in the same form as before:

$$L' = L + \frac{dF(q_1, ..., q_n, t)}{dt}$$

And if you understood the previous problem, you'll know why there is no effect on the motion of the particle(i.e. there are many Lagrangians that may describe the motion of a system, there is no unique Lagrangian).

10. Let $q_1, ..., q_n$ be a set of independent generalized coordinates for a system *of n degrees of freedom, with a Lagrangian $L(q, \dot{q}, t)$. Suppose we transform to another set of independent coordinates $s_1, ..., s_n$ by means of transformation equations*

$$q_i = q_i(s_1, ..., s_n, t), \qquad i = 1, ..., n.$$

(Such a transformatin is called a point transformation.) Show that if the Lagrangian function is expressed as a function of $s_j, \dot{s}_j$ and t through the equation of transformation, then L satisfies Lagrange's equations with respect to the s coordinates

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{s}_j} - \frac{\partial L}{\partial s_j} = 0$$

In other words, the form of Lagrange's equations is invariant under a point transformation.

Answer:

We know:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$

and we want to prove:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{s}_j} - \frac{\partial L}{\partial s_j} = 0$$

If we put $\frac{\partial L}{\partial \dot{s}_j}$ and $\frac{\partial L}{\partial s_j}$ in terms of the q coordinates, then they can be substitued back in and shown to still satisfy Lagrange's equations.

$$\frac{\partial L}{\partial s_j} = \sum_i \frac{\partial L}{\partial q_i}\frac{\partial q_i}{\partial s_j}$$

$$\frac{\partial L}{\partial \dot{s}_j} = \sum_i \frac{\partial L}{\partial \dot{q}_i}\frac{\partial \dot{q}_i}{\partial \dot{s}_j}$$

We know:

$$\frac{\partial q_i}{\partial s_j} = \frac{\partial \dot{q}_i}{\partial \dot{s}_j}$$

Thus,

$$\frac{\partial L}{\partial \dot{s}_j} = \sum_i \frac{\partial L}{\partial \dot{q}_i}\frac{\partial q_i}{\partial s_j}$$

Plug $\frac{\partial L}{\partial \dot{s}_j}$ and $\frac{\partial L}{\partial s_j}$ into the Lagrangian equation and see if they satisfy it:

$$\frac{d}{dt}[\sum_i \frac{\partial L}{\partial \dot{q}_i}\frac{\partial q_i}{\partial s_j}] - [\sum_i \frac{\partial L}{\partial q_i}\frac{\partial q_i}{\partial s_j}] = 0$$

Pulling out the summation to the right and $\frac{\partial q_i}{\partial s_j}$ to the left, we are left with:

$$\sum_i [\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i}]\frac{\partial q_i}{\partial s_j} = 0$$

This shows that Lagrangian's equations are invariant under a point transformation.

Goldstein Chapter 1 Exercises

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1 Exercises

11. Consider a uniform thin disk that rolls without slipping on a horizontal plane. A horizontal force is applied to the center of the disk and in a direction parallel to the plane of the disk.

- Derive Lagrange's equations and find the generalized force.
- Discuss the motion if the force is not applied parallel to the plane of the disk.

Answer:

To find Lagrangian's equations, we need to first find the Lagrangian.

$$L = T - V$$

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(r\omega)^2 \qquad V = 0$$

Therefore

$$L = \frac{1}{2}m(r\omega)^2$$

Plug into the Lagrange equations:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = Q$$

$$\frac{d}{dt}\frac{\partial \frac{1}{2}mr^2\omega^2}{\partial (r\omega)} - \frac{\partial \frac{1}{2}mr^2\omega^2}{\partial x} = Q$$

$$\frac{d}{dt}m(r\omega) = Q$$

$$m(r\ddot{\omega}) = Q$$

If the motion is not applied parallel to the plane of the disk, then there might be some slipping, or another generalized coordinate would have to be introduced, such as θ to describe the y-axis motion. The velocity of the disk would not just be in the x-direction as it is here.

12. The escape velocity of a particle on Earth is the minimum velocity required at Earth's surface in order that that particle can escape from Earth's gravitational field. Neglecting the resistance of the atmosphere, the system is conservative. From the conservation theorme for potential plus kinetic energy show that the escape veolcity for Earth, ingnoring the presence of the Moon, is 11.2 km/s.

Answer:

$$\frac{GMm}{r} = \frac{1}{2}mv^2$$
$$\frac{GM}{r} = \frac{1}{2}v^2$$

Lets plug in the numbers to this simple problem:

$$\frac{(6.67 \times 10^{-11}) \cdot (6 \times 10^{24})}{(6 \times 10^6)} = \frac{1}{2}v^2$$

This gives $v = 1.118 \times 10^4$ m/s which is 11.2 km/s.

13. Rockets are propelled by the momentum reaction of the exhaust gases expelled from the tail. Since these gases arise from the raction of the fuels carried in the rocket, the mass of the rocket is not constant, but decreases as the fuel is expended. Show that the equation of motion for a rocket projected vertically upward in a uniform gravitational field, neglecting atmospheric friction, is: $$m\frac{dv}{dt} = -v'\frac{dm}{dt} - mg$$ where m is the mass of the rocket and v' is the velocity of the escaping gases relative to the rocket. Integrate this equation to obtain v as a function of m, assuming a constant time rate of loss of mass. Show, for a rocket starting initally from rest, with v' equal to 2.1 km/s and a mass loss per second equal to 1/60th of the intial mass, that in order to reach the escape velocity the ratio of the wight of the fuel to the weight of the empty rocket must be almost 300!

Answer:

This problem can be tricky if you're not very careful with the notation. But here is the best way to do it. Defining m_e equal to the empty rocket mass, m_f is the total fuel mass, m_0 is the intitial rocket mass, that is, $m_e + m_f$, and $\frac{dm}{dt} = -\frac{m_0}{60}$ as the loss rate of mass, and finally the goal is to find the ratio of

m_f/m_e to be about 300.

The total force is just ma, as in Newton's second law. The total force on the rocket will be equal to the force due to the gas escaping minus the weight of the rocket:

$$ma = \frac{d}{dt}[-mv'] - mg$$

$$m\frac{dv}{dt} = -v'\frac{dm}{dt} - mg$$

The rate of lost mass is negative. The velocity is in the negative direction, so, with the two negative signs the term becomes positive.

Use this:

$$\frac{dv}{dm}\frac{dm}{dt} = \frac{dv}{dt}$$

Solve:

$$m\frac{dv}{dm}\frac{dm}{dt} = -v'\frac{dm}{dt} - mg$$

$$\frac{dv}{dm}\frac{dm}{dt} = -\frac{v'}{m}\frac{dm}{dt} - g$$

$$\frac{dv}{dm} = -\frac{v'}{m} + \frac{60g}{m_0}$$

Notice that the two negative signs cancelled out to give us a positive far right term.

$$dv = -\frac{v'}{m}dm + \frac{60g}{m_0}dm$$

Integrating,

$$\int dv = -v'\int_{m_0}^{m_e}\frac{dm}{m} + \int_{m_0}^{m_e}\frac{60g}{m_0}dm$$

$$v = -v'\ln\frac{m_e}{m_0} + \frac{60g}{m_0}(m_e - m_0)$$

$$v = -v'\ln\frac{m_e}{m_e + m_f} + 60g\frac{m_e - m_e - m_f}{m_e + m_f}$$

$$v = v'\ln\frac{m_e + m_f}{m_e} - 60g\frac{m_f}{m_e + m_f}$$

Now watch this, I'm going to use my magic wand of approximation. This is when I say that because I know that the ratio is so big, I can ignore the empty rocket mass as compared to the fuel mass. $m_e << m_f$. Let me remind you, we are looking for this ratio as well. The ratio of the fuel mass to empty rocket, m_f/m_e.

$$v = v' \ln \frac{m_e + m_f}{m_e} - 60g\frac{m_f}{m_e + m_f}$$

$$v = v' \ln \frac{m_f}{m_e} - 60g\frac{m_f}{m_f}$$

$$\frac{v + 60g}{v'} = \ln \frac{m_f}{m_e}$$

$$\exp[\frac{v + 60g}{v'}] = \frac{m_f}{m_e}$$

Plug in 11,200 m/s for v, 9.8 for g, and 2100 m/s for v'.

$$\frac{m_f}{m_e} = 274$$

And, by the way, if Goldstein hadn't just converted 6800 ft/s from his second edition to 2.1 km/s in his third edition without checking his answer, he would have noticed that 2.07 km/s which is a more accurate approximation, yields a ratio of 296. This is more like the number 300 he was looking for.

14. Two points of mass m are joined by a rigid weightless rod of length l, the center of which is constrained to move on a circle of radius a. Express the kinetic energy in generalized coordinates.

Answer:

$$T_1 + T_2 = T$$

Where T_1 equals the kinetic energy of the center of mass, and T_2 is the kinetic energy about the center of mass. Keep these two parts seperate!

Solve for T_1 first, its the easiest:

$$T_1 = \frac{1}{2}Mv_{cm}^2 = \frac{1}{2}(2m)(a\dot{\psi})^2 = ma^2\dot{\psi}^2$$

Solve for T_2, realizing that the rigid rod is not restricted to just the X-Y plane. Don't forget the Z-axis!

$$T_2 = \frac{1}{2}Mv^2 = mv^2$$

Solve for v^2 about the center of mass. The angle ϕ will be the angle in the x-y plane, while the angle θ will be the angle from the z-axis.

If $\theta = 90^o$ and $\phi = 0^o$ then $x = l/2$ so:

$$x = \frac{l}{2}\sin\theta\cos\phi$$

If $\theta = 90^o$ and $\phi = 90^o$ then $y = l/2$ so:

$$y = \frac{l}{2}\sin\theta\sin\phi$$

If $\theta = 0^o$, then $z = l/2$ so:

$$z = \frac{l}{2}\cos\theta$$

Find v^2:

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = v^2$$

$$\dot{x} = \frac{l}{2}(\cos\phi\cos\theta\dot{\theta} - \sin\theta\sin\phi\dot{\phi})$$

$$\dot{y} = \frac{1}{2}(\sin\phi\cos\theta\dot{\theta} + \sin\theta\cos\phi\dot{\phi})$$

$$\dot{z} = -\frac{l}{2}\sin\theta\dot{\theta}$$

Carefully square each:

$$\dot{x}^2 = \frac{l^2}{4}\cos^2\phi\cos^2\theta\dot{\theta}^2 - 2\frac{l}{2}\sin\theta\sin\phi\dot{\phi}\frac{l}{2}\cos\phi\cos\theta\dot{\theta} + \frac{l^2}{4}\sin^2\theta\sin^2\phi\dot{\phi}^2$$

$$\dot{y}^2 = \frac{l^2}{4}\sin^2\phi\cos^2\theta\dot{\theta}^2 + 2\frac{l}{2}\sin\theta\cos\phi\dot{\phi}\frac{l}{2}\sin\phi\cos\theta\dot{\theta} + \frac{l^2}{4}\sin^2\theta\cos^2\phi\dot{\phi}^2$$

$$\dot{z}^2 = \frac{l^2}{4}\sin^2\theta\dot{\theta}^2$$

Now add, striking out the middle terms:

$$\dot{x}^2+\dot{y}^2+\dot{z}^2 = \frac{l^2}{4}[\cos^2\phi\cos^2\theta\dot{\theta}^2+\sin^2\theta\sin^2\phi\dot{\phi}^2+\sin^2\phi\cos^2\theta\dot{\theta}^2+\sin^2\theta\cos^2\phi\dot{\phi}^2+\sin^2\theta\dot{\theta}^2]$$

Pull the first and third terms inside the brackets together, and pull the second and fourth terms together as well:

$$v^2 = \frac{l^2}{4}[\cos^2\theta\dot{\theta}^2(\cos^2\phi + \sin^2\phi) + \sin^2\theta\dot{\phi}^2(\sin^2\phi + \cos^2\phi) + \sin^2\theta\dot{\theta}^2]$$

$$v^2 = \frac{l^2}{4}(\cos^2\theta\dot{\theta}^2 + \sin^2\theta\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2)$$

$$v^2 = \frac{l^2}{4}(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2)$$

Now that we finally have v^2 we can plug this into T_2

$$T = T_1 + T_2 = ma^2\dot{\psi}^2 + m\frac{l^2}{4}(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2)$$

It was important to emphasize that T_1 is the kinetic energy of the total mass around the center of the circle while T_2 is the kinetic energy of the masses about the center of mass. Hope that helped.

15. A point particle moves in space under the influence of a force derivable from a generalized potential of the form

$$U(\mathbf{r}, \mathbf{v}) = V(r) + \sigma \cdot \mathbf{L}$$

where r is the radius vector from a fixed point, L is the angular momentum about that point, and σ is a fixed vector in space.

1. Find the components of the force on the particle in both Cartesian and spherical poloar coordinates, on the basis of Lagrangian's equations with a generalized potential
2. Show that the components in the two coordinate systems are related to each other as in the equation shown below of generalized force
3. Obtain the equations of motion in spherical polar coordinates

$$Q_j = \sum_i \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j}$$

Answer:

This one is a fairly tedious problem mathematically. First lets find the components of the force in Cartesian coordinates. Convert $U(r, v)$ into Cartesian and then plug the expression into the Lagrange-Euler equation.

$$Q_j = \frac{d}{dt}\frac{\partial}{\partial \dot{q}_j}[V(\sqrt{x^2+y^2+z^2}) + \sigma\cdot(r\times p)] - \frac{\partial}{\partial q_j}[V(\sqrt{x^2+y^2+z^2}) + \sigma\cdot(r\times p)]$$

$$Q_j = \frac{d}{dt}\frac{\partial}{\partial \dot{v}_j}[\sigma\cdot[(x\hat{i}+y\hat{j}+z\hat{k})\times(p_x\hat{i}+p_y\hat{j}+p_z\hat{k})] - \frac{\partial}{\partial x_j}[V(\sqrt{x^2+y^2+z^2}) + \sigma\cdot(r\times p)]$$

$$Q_j = \frac{d}{dt}\frac{\partial}{\partial \dot{v}_j}[\sigma \cdot [(yp_z - zp_y)\hat{i} + (zp_x - xp_z)\hat{j} + (xp_y - p_x y)\hat{k}] - \frac{\partial}{\partial x_j}[V(\sqrt{x^2+y^2+z^2}) + \sigma \cdot (r \times p)]$$

$$Q_j = \frac{d}{dt}\frac{\partial}{\partial \dot{v}_j}[m\sigma_x(yv_z - zv_y) + m\sigma_y(zv_x - xv_z) + m\sigma_z(xv_y - v_x y)] - \frac{\partial}{\partial x_j}[V(\sqrt{x^2+y^2+z^2}) + \sigma \cdot (r \times p)]$$

Where we know that

$$m\sigma_x(yv_z - zv_y) + m\sigma_y(zv_x - xv_z) + m\sigma_z(xv_y - v_x y) = \sigma \cdot (r \times p)$$

So lets solve for just one component first and let the other ones follow by example:

$$Q_x = \frac{d}{dt}(m\sigma_y z - m\sigma_z y) - \frac{\partial}{\partial x}[V(\sqrt{x^2+y^2+z^2}) + m\sigma_x(yv_z - zv_y) + m\sigma_y(zv_x - xv_z) + m\sigma_z(xv_y - v_x y)]$$

$$Q_x = m(\sigma_y v_z - \sigma_z v_y) - [V'(\sqrt{x^2+y^2+z^2})(x^2+y^2+z^2)^{-\frac{1}{2}}x - m\sigma_y v_z + m\sigma_z v_y]$$

$$Q_x = 2m(\sigma_y v_z - \sigma_z v_y) - V'\frac{x}{r}$$

If you do the same for the y and z components, they are:

$$Q_y = 2m(\sigma_z v_x - \sigma_x v_z) - V'\frac{y}{r}$$

$$Q_z = 2m(\sigma_x v_y - \sigma_y v_x) - V'\frac{z}{r}$$

Thus the generalized force is:

$$F = 2m(\sigma \times \mathbf{v}) - V'\frac{\mathbf{r}}{r}$$

Now its time to play with spherical coordinates. The trick to this is setting up the coordinate system so that σ is along the z axis. Thus the dot product simplifies and L is only the z-component.

$$U = V(r) + m\sigma(x\dot{y} - y\dot{x})$$

With spherical coordinate definitions:

$$x = r\sin\theta\cos\phi \quad y = r\sin\theta\sin\phi \quad z = r\cos\theta$$

Solving for $(x\dot{y} - y\dot{x})$

$$\dot{x} = r(-\sin\theta\sin\phi\dot{\phi} + \cos\phi\cos\theta\dot{\theta}) + \dot{r}\sin\theta\cos\phi$$
$$\dot{y} = r(\sin\theta\cos\phi\dot{\phi} + \sin\phi\cos\theta\dot{\theta}) + \dot{r}\sin\theta\sin\phi$$

Thus $x\dot{y} - y\dot{x}$ is

$$= r\sin\theta\cos\phi[r(\sin\theta\cos\phi\dot{\phi} + \sin\phi\cos\theta\dot{\theta}) + \dot{r}\sin\theta\sin\phi]$$
$$-r\sin\theta\sin\phi[r(-\sin\theta\sin\phi\dot{\phi} + \cos\phi\cos\theta\dot{\theta}) + \dot{r}\sin\theta\cos\phi].$$

Note that the $\dot{r}$ terms drop out as well as the $\dot{\theta}$ terms.

$$x\dot{y} - y\dot{x} = r^2\sin^2\theta\cos^2\phi\dot{\phi} + r^2\sin^2\theta\sin^2\phi\dot{\phi}$$
$$x\dot{y} - y\dot{x} = r^2\sin^2\theta\dot{\phi}$$

Thus

$$U = V(r) + m\sigma r^2\sin^2\theta\dot{\phi}$$

Plugging this in to Lagrangian's equations yields:

For Q_r:

$$Q_r = -\frac{\partial U}{\partial r} + \frac{d}{dt}(\frac{\partial U}{\partial \dot{r}})$$
$$Q_r = -\frac{dV}{dr} - 2m\sigma r\sin^2\theta\dot{\phi} + \frac{d}{dt}(0)$$
$$Q_r = -\frac{dV}{dr} - 2m\sigma r\sin^2\theta\dot{\phi}$$

For Q_θ:

$$Q_\theta = -2m\sigma r^2\sin\theta\dot{\phi}\cos\theta$$

For Q_ϕ:

$$Q_\phi = \frac{d}{dt}(m\sigma r^2\sin^2\theta)$$
$$Q_\phi = m\sigma(r^2 2\sin\theta\cos\theta\dot{\theta} + \sin^2\theta 2r\dot{r})$$
$$Q_\phi = 2m\sigma r^2\sin\theta\cos\theta\dot{\theta} + 2m\sigma r\dot{r}\sin^2\theta$$

For part b, we have to show the components of the two coordinate systems are related to each other via

$$Q_j = \sum_i \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j}$$

Lets take ϕ for an example,

$$Q_\phi = \mathbf{F} \cdot \frac{\partial \mathbf{r}}{\partial \phi} = F_x\frac{\partial x}{\partial \phi} + F_y\frac{\partial y}{\partial \phi} + F_z\frac{\partial z}{\partial \phi}$$

$$Q_\phi = Q_x(-r\sin\theta\sin\phi) + Q_y(r\sin\theta\cos\phi) + Q_z(0)$$

$$Q_\phi = [2m(\sigma_y v_z - \sigma_z v_y) - V'\frac{x}{r}](-r\sin\theta\sin\phi) + [2m(\sigma_z v_x - \sigma_x v_z) - V'\frac{y}{r}](r\sin\theta\cos\phi) + 0$$

Because in both coordinate systems we will have σ pointing in only the z direction, then the x and y σ's disappear:

$$Q_\phi = [2m(-\sigma_z v_y) - V'\frac{x}{r}](-r\sin\theta\sin\phi) + [2m(\sigma_z v_x) - V'\frac{y}{r}](r\sin\theta\cos\phi)$$

Pull out the V' terms, plug in x and y, see how V' terms cancel

$$Q_\phi = V'(x\sin\theta\sin\phi - y\sin\theta\cos\phi) - 2mr\sin\theta\sigma[v_y\sin\phi + v_x\cos\phi]$$

$$Q_\phi = V'(r\sin^2\theta\cos\phi\sin\phi - r\sin^2\theta\sin\phi\cos\phi) - 2mr\sin\theta\sigma[v_y\sin\phi + v_x\cos\phi]$$

$$Q_\phi = -2mr\sin\theta\sigma[v_y\sin\phi + v_x\cos\phi]$$

Plug in v_y and v_x:

$$Q_\phi = -2mr\sin\theta\sigma[\sin\phi(r\sin\theta\cos\phi\dot{\phi} + r\sin\phi\cos\theta\dot{\theta} + \dot{r}\sin\theta\sin\phi)$$

$$+\cos\phi(-r\sin\theta\sin\phi\dot{\phi} + r\cos\phi\cos\theta\dot{\theta} + \dot{r}\sin\theta\cos\phi)].$$

$$Q_\phi = 2m\sigma r\sin\theta[r\sin^2\phi\cos\theta\dot{\theta} + \dot{r}\sin\theta\sin^2\phi$$
$$+r\cos^2\phi\cos\theta\dot{\theta} + \dot{r}\sin\theta\cos^2\phi].$$

Gather sin^2's and cos^2's:

$$Q_\phi = 2m\sigma r\sin\theta[r\cos\theta\dot{\theta} + \dot{r}\sin\theta]$$

This checks with the derivation in part a for Q_ϕ. This shows that indeed the components in the two coordinate systems are related to each other as

$$Q_j = \sum_i \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j}$$

Any of the other components could be equally compared in the same procedure. I chose Q_ϕ because I felt it was easiest to write up.

For part c, to obtain the equations of motion, we need to find the generalized kinetic energy. From this we'll use Lagrange's equations to solve for each component of the force. With both derivations, the components derived from the generalized potential, and the components derived from kinetic energy, they will be set equal to each other.

In spherical coordinates, v is:

$$\mathbf{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\sin\theta\dot{\phi}\hat{\phi}$$

The kinetic energy in spherical polar coordinates is then:

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2)$$

For the r component:

$$\frac{d}{dt}(\frac{\partial T}{\partial \dot{r}}) - \frac{\partial T}{\partial r} = Q_r$$

$$\frac{d}{dt}(m\dot{r}) - mr\dot{\theta}^2 - mr\sin^2\theta\dot{\phi}^2 = Q_r$$

$$m\ddot{r} - mr\dot{\theta}^2 - mr\sin^2\theta\dot{\phi}^2 = Q_r$$

From part a,

$$Q_r = -V' - 2m\sigma r\sin^2\theta\dot{\phi}$$

Set them equal:

$$m\ddot{r} - mr\dot{\theta}^2 - mr\sin^2\theta\dot{\phi}^2 = Q_r = -V' - 2m\sigma r\sin^2\theta\dot{\phi}$$

$$m\ddot{r} - mr\dot{\theta}^2 - mr\sin^2\theta\dot{\phi}^2 + V' + 2m\sigma r\sin^2\theta\dot{\phi} = 0$$

$$m\ddot{r} - mr\dot{\theta}^2 + mr\sin^2\theta\dot{\phi}(2\sigma - \dot{\phi}) + V' = 0$$

For the θ component:

$$\frac{d}{dt}(\frac{\partial T}{\partial \dot{\theta}}) - \frac{\partial T}{\partial \theta} = Q_\theta$$

$$\frac{d}{dt}(mr^2\dot{\theta}) - mr^2\sin\theta\dot{\phi}^2\cos\theta = Q_\theta$$

$$mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} - mr^2\sin\theta\dot{\phi}^2\cos\theta = Q_\theta$$

From part a,

$$Q_\theta = -2m\sigma r^2\sin\theta\cos\theta\dot{\phi}$$

Set the two equal:

$$mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} - mr^2\sin\theta\dot{\phi}^2\cos\theta + 2m\sigma r^2\sin\theta\cos\theta\dot{\phi} = 0$$

$$mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} + mr^2\sin\theta\cos\theta\dot{\phi}(2\sigma - \dot{\phi}) = 0$$

For the last component, ϕ we have:

$$\frac{d}{dt}(\frac{\partial T}{\partial \dot{\phi}}) - \frac{\partial T}{\partial \phi} = Q_\phi$$

$$\frac{d}{dt}(mr^2 \sin^2 \theta \dot{\phi}) - 0 = Q_\phi$$

$$mr^2 \frac{d}{dt}(\sin^2 \theta \dot{\phi}) + 2mr\dot{r} \sin^2 \theta \dot{\phi} = Q_\phi$$

$$mr^2 \sin^2 \theta \ddot{\phi} + 2mr^2 \sin \theta \cos \theta \dot{\theta} \dot{\phi} + 2mr\dot{r} \sin^2 \theta \dot{\phi} = Q_\phi$$

From part a,

$$Q_\phi = 2m\sigma r^2 \sin \theta \cos \theta \dot{\theta} + 2m\sigma r\dot{r} \sin^2 \theta$$

Set the two equal:

$$mr^2 \sin^2 \theta \ddot{\phi} + 2mr^2 \sin \theta \cos \theta \dot{\theta} \dot{\phi} + 2mr\dot{r} \sin^2 \theta \dot{\phi} - 2m\sigma r^2 \sin \theta \cos \theta \dot{\theta} - 2m\sigma r\dot{r} \sin^2 \theta = 0$$

$$mr^2 \sin^2 \theta \ddot{\phi} + 2mr^2 \sin \theta \cos \theta \dot{\theta}(\dot{\phi} - \sigma) + 2mr\dot{r} \sin^2 \theta(\dot{\phi} - \sigma) = 0$$

That's it, here are all of the equations of motion together in one place:

$$m\ddot{r} - mr\dot{\theta}^2 + mr \sin^2 \theta \dot{\phi}(2\sigma - \dot{\phi}) + V' = 0$$
$$mr^2 \ddot{\theta} + 2mr\dot{r}\dot{\theta} + mr^2 \sin \theta \cos \theta \dot{\phi}(2\sigma - \dot{\phi}) = 0$$
$$mr^2 \sin^2 \theta \ddot{\phi} + 2mr^2 \sin \theta \cos \theta \dot{\theta}(\dot{\phi} - \sigma) + 2mr\dot{r} \sin^2 \theta(\dot{\phi} - \sigma) = 0$$

16. A particle moves in a plane under the influence of a force, acting toward a center of force, whose magnitude is

$$F = \frac{1}{r^2}(1 - \frac{\dot{r}^2 - 2\ddot{r}r}{c^2})$$

where r is the distance of the particle to the center of force. Find the generalized potential that will result in such a force, and from that the Lagrangian for the motion in a plane. The expression for F represents the force between two charges in Weber's electrodynamics.

Answer:

This one takes some guess work and careful handling of signs. To get from force to potential we will have to take a derivative of a likely potential. Note that if you expand the force it looks like this:

$$F = \frac{1}{r^2} - \frac{\dot{r}^2}{c^2 r^2} + \frac{2\ddot{r}}{c^2 r}$$

We know that

$$-\frac{\partial U}{\partial r} + \frac{d}{dt}\frac{\partial U}{\partial \dot{r}} = F$$

So lets focus on the time derivative for now. If we want a $\ddot{r}$ we would have to take the derivative of a $\dot{r}$. Let pick something that looks close, say $\frac{2\dot{r}}{c^2 r}$:

$$\frac{d}{dt}(\frac{2\dot{r}}{c^2 r}) = \frac{2\dot{r}}{c^2}(-\frac{\dot{r}}{r^2}) + \frac{2\ddot{r}}{c^2 r} = -\frac{2\dot{r}^2}{c^2 r^2} + \frac{2\ddot{r}}{c^2 r}$$

Excellent! This has our third term we were looking for. Make this stay the same when you take the partial with respect to $\dot{r}$.

$$\frac{\partial}{\partial \dot{r}}\frac{\dot{r}^2}{c^2 r} = \frac{2\dot{r}}{c^2 r}$$

So we know that the potential we are guessing at, has the term $\frac{\dot{r}^2}{c^2 r}$ in it. Lets add to it what would make the first term of the force if you took the negative partial with respect to r, see if it works out.

That is,

$$-\frac{\partial}{\partial r}\frac{1}{r} = \frac{1}{r^2}$$

So

$$U = \frac{1}{r} + \frac{\dot{r}^2}{c^2 r}$$

might work. Checking:

$$-\frac{\partial U}{\partial r} + \frac{d}{dt}\frac{\partial U}{\partial \dot{r}} = F$$

We have

$$\frac{\partial U}{\partial r} = -\frac{1}{r^2} - \frac{\dot{r}^2}{c^2 r^2}$$

and

$$\frac{d}{dt}\frac{\partial U}{\partial \dot{r}} = \frac{d}{dt}\frac{2\dot{r}}{c^2 r} = \frac{2\dot{r}}{c^2}(-\frac{\dot{r}}{r^2}) + \frac{1}{r}(\frac{2\ddot{r}}{c^2}) = -\frac{2\dot{r}^2}{c^2 r^2} + \frac{2\ddot{r}}{c^2 r}$$

thus

$$-\frac{\partial U}{\partial r} + \frac{d}{dt}\frac{\partial U}{\partial \dot{r}} = \frac{1}{r^2} + \frac{\dot{r}^2}{c^2 r^2} - \frac{2\dot{r}^2}{c^2 r^2} + \frac{2\ddot{r}}{c^2 r}$$

$$-\frac{\partial U}{\partial r} + \frac{d}{dt}\frac{\partial U}{\partial \dot{r}} = \frac{1}{r^2} - \frac{\dot{r}^2}{c^2 r^2} + \frac{2\ddot{r}}{c^2 r}$$

This is indeed the force unexpanded,

$$F = \frac{1}{r^2}(1 - \frac{\dot{r}^2 - 2\ddot{r}r}{c^2}) = \frac{1}{r^2} - \frac{\dot{r}^2}{c^2 r^2} + \frac{2\ddot{r}}{c^2 r}$$

Thus our potential, $U = \frac{1}{r} + \frac{\dot{r}^2}{c^2 r}$ works. To find the Lagrangian use $L = T - U$. In a plane, with spherical coordinates, the kinetic energy is

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$$

Thus

$$L = \frac{1}{2}m(\dot{r}^2 + \dot{r}^2\dot{\theta}^2) - \frac{1}{r}(1 + \frac{\dot{r}^2}{c^2})$$

17. A nucleus, originally at rest, decays radioactively by emitting an electron of momentum 1.73 MeV/c, and at right angles to the direction of the electron a neutrino with momentum 1.00 MeV/c. The MeV, million electron volt, is a unit of energy used in modern physics equal to 1.60×10^{-13} J. Correspondingly, MeV/c is a unit of linear momentum equal to 5.34×10^{-22} kg·m/s. In what direction does the nucleus recoil? What is its momentum in MeV/c? If the mass of the residual nucleus is 3.90×10^{-25} kg what is its kinetic energy, in electron volts?

Answer:

If you draw a diagram you'll see that the nucleus recoils in the opposite direction of the vector made by the electron plus the neutrino emission. Place the neutrino at the x-axis, the electron on the y axis and use pythagorean's theorme to see the nucleus will recoil with a momentum of 2 Mev/c. The nucleus goes in the opposite direction of the vector that makes an angle

$$\theta = \tan^{-1}\frac{1.73}{1} = 60°$$

from the x axis. This is 240° from the x-axis.

To find the kinetic energy, you can convert the momentum to $kg \cdot m/s$, then convert the whole answer that is in joules to eV,

$$T = \frac{p^2}{2m} = \frac{[2(5.34 \times 10^{-22})]^2}{2 \cdot 3.9 \times 10^{-25}} \cdot \frac{1MeV}{1.6 \times 10^{-13}J} \cdot \frac{10^6 eV}{1MeV} = 9.13eV$$

18. A Lagrangian for a particular physical system can be written as

$$L' = \frac{m}{2}(a\dot{x}^2 + 2b\dot{x}\dot{y} + c\dot{y}^2) - \frac{K}{2}(ax^2 + 2bxy + cy^2).$$

where a, b, and c are arbitrary constants but subject to the condition that $b^2 - ac \neq 0$. What are the equations of motion? Examine particularly the two cases $a = 0 = c$ and $b = 0$, $c = -a$. What is the physical system described by the above Lagrangian? Show that the usual Lagrangian for this system as defined by Eq. (1.57'):

$$L'(q, \dot{q}, t) = L(q, \dot{q}, t) + \frac{dF}{dt}$$

is related to L' by a point transformation (cf. Derivation 10). What is the significance of the condition on the value of $b^2 - ac$?

Answer:

To find the equations of motion, use the Euler-Lagrange equations.

$$\frac{\partial L'}{\partial q} = \frac{d}{dt}\frac{\partial L'}{\partial \dot{q}}$$

For x first:

$$-\frac{\partial L'}{\partial x} = -(-Kax - Kby) = K(ax + by)$$

$$\frac{\partial L'}{\partial \dot{x}} = m(a\dot{x} + b\dot{y})$$

$$\frac{d}{dt}\frac{\partial L'}{\partial \dot{x}} = m(a\ddot{x} + b\ddot{y})$$

Thus

$$-K(ax + by) = m(a\ddot{x} + b\ddot{y})$$

Now for y:

$$-\frac{\partial L'}{\partial y} = -(-Kby - Kcy) = K(bx + cy)$$

$$\frac{\partial L'}{\partial \dot{x}} = m(b\dot{x} + c\dot{y})$$

$$\frac{d}{dt}\frac{\partial L'}{\partial \dot{x}} = m(b\ddot{x} + c\ddot{y})$$

Thus

$$-K(bx + cy) = m(b\ddot{x} + c\ddot{y})$$

Therefore our equations of motion are:

$$-K(ax+by) = m(a\ddot{x}+b\ddot{y})$$
$$-K(bx+cy) = m(b\ddot{x}+c\ddot{y})$$

Examining the particular cases, we find:

If $a = 0 = c$ then:

$$-Kx = m\ddot{x} \quad -Ky = -m\ddot{y}$$

If $b = 0$, $c = -a$ then:

$$-Kx = m\ddot{x} \quad -Ky = -m\ddot{y}$$

The physical system is harmonic oscillation of a particle of mass m in two dimensions. If you make a substitution to go to a different coordinate system this is easier to see.

$$u = ax+by \quad v = bx+cy$$

Then

$$-Ku = m\ddot{u}$$
$$-Kv = m\ddot{v}$$

The system can now be more easily seen as two independent but identical simple harmonic oscillators, after a point transformation was made.

When the condition $b^2 - ac \neq 0$ is violated, then we have $b = \sqrt{ac}$, and L' simplifies to this:

$$L' = \frac{m}{2}(\sqrt{a}\dot{x}+\sqrt{c}\dot{y})^2 - \frac{K}{2}(\sqrt{a}x+\sqrt{c}y)^2$$

Note that this is now a one dimensional problem. So the condition keeps the Lagrangian in two dimensions, or you can say that the transformation matrix

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

is singluar because $b^2 - ac \neq 0$ Note that

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

So if this condition holds then we can reduce the Lagrangian by a point transformation.

$\boxed{\text{19. Obtain the Lagrange equations of motion for spherical pendulum, i.e., a mass point suspended by a rigid weightless rod.}}$

Answer:

The kinetic energy is found the same way as in exercise 14, and the potential energy is found by using the origin to be at zero potential.

$$T = \frac{1}{2}ml^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2)$$

If θ is the angle from the positive z-axis, then at $\theta = 90°$ the rod is aligned along the x-y plane, with zero potential. Because $cos(90) = 0$ we should expect a cos in the potential. When the rod is aligned along the z-axis, its potential will be its height.

$$V = mgl\cos\theta$$

If $\theta = 0$ then $V = mgl$. If $\theta = 180$ then $V = -mgl$.
So the Lagrangian is $L = T - V$.

$$L = \frac{1}{2}ml^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) - mgl\cos\theta$$

To find the Lagrangian equations, they are the equations of motion:

$$\frac{\partial L}{\partial \theta} = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}}$$

$$\frac{\partial L}{\partial \phi} = \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}}$$

Solving these yields:

$$\frac{\partial L}{\partial \theta} = ml^2\sin\theta\dot{\phi}^2\cos\theta + mgl\sin\theta$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = ml^2\ddot{\theta}$$

Thus

$$ml^2\sin\theta\dot{\phi}^2\cos\theta + mgl\sin\theta - ml^2\ddot{\theta} = 0$$

and

$$\frac{\partial L}{\partial \phi} = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = \frac{d}{dt}(ml^2\sin^2\theta\dot{\phi}) = ml^2\sin^2\theta\ddot{\phi} + 2\dot{\phi}ml^2\sin\theta\cos\theta$$

Thus

$$ml^2\sin^2\theta\ddot{\phi} + 2\dot{\phi}ml^2\sin\theta\cos\theta = 0$$

Therefore the equations of motion are:

$$ml^2 \sin\theta\dot{\phi}^2 \cos\theta + mgl \sin\theta - ml^2\ddot{\theta} = 0$$

$$ml^2 \sin^2\theta\ddot{\phi} + 2\dot{\phi}ml^2 \sin\theta\cos\theta = 0$$

20. A particle of mass m moves in one dimension such that it has the Lagrangian

$$L = \frac{m^2\dot{x}^4}{12} + m\dot{x}^2 V(x) - V_2(x)$$

where V is some differentiable function of x. Find the equation of motion for $x(t)$ and describe the physical nature of the system on the basis of this system.

Answer:

I believe there are two errors in the 3rd edition version of this question. Namely, there should be a negative sign infront of $m\dot{x}^2V(x)$ and the $V_2(x)$ should be a $V^2(x)$. Assuming these are all the errors, the solution to this problem goes like this:

$$L = \frac{m^2\dot{x}^4}{12} - m\dot{x}^2 V(x) - V^2(x)$$

Find the equations of motion from Euler-Lagrange formulation.

$$\frac{\partial L}{\partial x} = -m\dot{x}^2V'(x) - 2V(x)V'(x)$$

$$\frac{\partial L}{\partial \dot{x}} = \frac{m^2\dot{x}^3}{3} + 2m\dot{x}V(x)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = m^2\dot{x}^2\ddot{x} + 2mV(x)\ddot{x}$$

Thus

$$m\dot{x}^2V' + 2VV' + m^2\dot{x}^2\ddot{x} + 2mV\ddot{x} = 0$$

is our equation of motion. But we want to interpret it. So lets make it look like it has useful terms in it, like kinetic energy and force. This can be done by dividing by 2 and seperating out $\frac{1}{2}mv^2$ and ma's.

$$\frac{m\dot{x}^2}{2}V' + VV' + \frac{m\dot{x}^2}{2}m\ddot{x} + m\ddot{x}V = 0$$

Pull V' terms together and $m\ddot{x}$ terms together:

$$(\frac{m\dot{x}^2}{2} + V)V' + m\ddot{x}(\frac{m\dot{x}^2}{2} + V) = 0$$

Therefore:

$$(\frac{m\dot{x}^2}{2} + V)(m\ddot{x} + V') = 0$$

Now this looks like $E \cdot E' = 0$ because $E = \frac{m\dot{x}^2}{2} + V(x)$. That would mean

$$\frac{d}{dt}E^2 = 2EE' = 0$$

Which allows us to see that E^2 is a constant. If you look at $t = 0$ and the starting energy of the particle, then you will notice that if $E = 0$ at $t = 0$ then $E = 0$ for all other times. If $E \neq 0$ at $t = 0$ then $E \neq 0$ all other times while $m\ddot{x} + V' = 0$.

21. Two mass points of mass m_1 and m_2 are connected by a string passing through a hole in a smooth table so that m_1 rests on the table surface and m_2 hangs suspended. Assuming m_2 moves only in a vertical line, what are the generalized coordinates for the system? Write the Lagrange equations for the system and, if possible, discuss the physical significance any of them might have. Reduce the problem to a single second-order differential equation and obtain a first integral of the equation. What is its physical significance? (Consider the motion only until m_1 reaches the hole.)

Answer:

The generalized coordinates for the system are θ, the angle m_1 moves round on the table, and r the length of the string from the hole to m_1. The whole motion of the system can be described by just these coordinates. To write the Lagrangian, we will want the kinetic and potential energies.

$$T = \frac{1}{2}m_2\dot{r}^2 + \frac{1}{2}m_1(\dot{r}^2 + r^2\dot{\theta}^2)$$

$$V = -m_2 g(l - r)$$

The kinetic energy is just the addition of both masses, while V is obtained so that $V = -mgl$ when $r = 0$ and so that $V = 0$ when $r = l$.

$$L = T - V = \frac{1}{2}(m_2 + m_1)\dot{r}^2 + \frac{1}{2}m_1 r^2\dot{\theta}^2 + m_2 g(l - r)$$

To find the Lagrangian equations or equations of motion, solve for each component:

$$\frac{\partial L}{\partial \theta} = 0$$

$$\frac{\partial L}{\partial \dot{\theta}} = m_1 r^2 \dot{\theta}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{d}{dt}(m_1 r^2 \dot{\theta}) = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = m_1 r^2 \ddot{\theta} + 2m_1 r\dot{r}\dot{\theta} = 0$$

Thus

$$\frac{d}{dt}(m_1 r^2 \dot{\theta}) = m_1 r(r\ddot{\theta} + 2\dot{\theta}\dot{r}) = 0$$

and

$$\frac{\partial L}{\partial r} = -m_2 g + m_1 r\dot{\theta}^2$$

$$\frac{\partial L}{\partial \dot{r}} = (m_2 + m_1)\dot{r}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = (m_2 + m_1)\ddot{r}$$

Thus

$$m_2 g - m_1 r\dot{\theta}^2 + (m_2 + m_1)\ddot{r} = 0$$

Therefore our equations of motion are:

$$\frac{d}{dt}(m_1 r^2 \dot{\theta}) = m_1 r(r\ddot{\theta} + 2\dot{\theta}\dot{r}) = 0$$

$$m_2 g - m_1 r\dot{\theta}^2 + (m_2 + m_1)\ddot{r} = 0$$

See that $m_1 r^2 \dot{\theta}$ is constant. It is angular momentum. Now the Lagrangian can be put in terms of angular momentum. We have $\dot{\theta} = l/m_1 r^2$.

$$L = \frac{1}{2}(m_1 + m_2)\dot{r}^2 + \frac{l^2}{2m_1 r^2} - m_2 g r$$

The equation of motion

$$m_2 g - m_1 r\dot{\theta}^2 + (m_2 + m_1)\ddot{r} = 0$$

Becomes

$$(m_1 + m_2)\ddot{r} - \frac{l^2}{m_1 r^3} + m_2 g = 0$$

The problem has been reduced to a single second-order differential equation. The next step is a nice one to notice. If you take the derivative of our new Lagrangian you get our single second-order differential equation of motion.

$$\frac{d}{dt}(\frac{1}{2}(m_1 + m_2)\dot{r}^2 + \frac{l^2}{2m_1 r^2} - m_2 g r) = (m_1 + m_2)\dot{r}\ddot{r} - \frac{l^2}{m_1 r^3}\dot{r} - m_2 g\dot{r} = 0$$

$$(m_1 + m_2)\ddot{r} - \frac{l^2}{m_1 r^3} - m_2 g = 0$$

Thus the first integral of the equation is exactly the Lagrangian. As far as interpreting this, I will venture to say the the Lagrangian is constant, the system is closed, the energy is conversed, the linear and angular momentum are conserved.

22. Obtain the Lagrangian and equations of motion for the double pendulum illustrated in Fig 1.4, where the lengths of the pendula are l_1 and l_2 with corresponding masses m_1 and m_2.

Answer:

Add the Lagrangian of the first mass to the Lagrangian of the second mass. For the first mass:

$$T_1 = \frac{1}{2} m l_1^2 \dot{\theta}_1^2$$

$$V_1 = -m_1 g l_1 \cos\theta_1$$

Thus

$$L_1 = T_1 - V_1 = \frac{1}{2} m l_1 \dot{\theta}_1^2 + m g l_1 \cos\theta_1$$

To find the Lagrangian for the second mass, use new coordinates:

$$x_2 = l_1 \sin\theta_1 + l_2 \sin\theta_2$$
$$y_2 = l_1 \cos\theta_1 + l_2 \cos\theta_2$$

Then it becomes easier to see the kinetic and potential energies:

$$T_2 = \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$$

$$V_2 = -m_2 g y_2$$

Take derivatives and then plug and chug:

$$T_2 = \frac{1}{2} m_2 (l_1^2 \sin^2\theta_1 \dot{\theta}_1^2 + 2 l_1 l_2 \sin\theta_1 \sin\theta_2 \dot{\theta}_1 \dot{\theta}_2 + l_2^2 \sin^2\theta_2 \dot{\theta}_2^2$$
$$+ l_1^2 \cos^2\theta_1 \dot{\theta}_1^2 + 2 l_1 l_2 \cos\theta_1 \cos\theta_2 \dot{\theta}_1 \dot{\theta}_2 + l_2^2 \cos^2\theta_2 \dot{\theta}_2^2)$$

$$T_2 = \frac{1}{2} m_2 (l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \cos(\theta_1 - \theta_2) \dot{\theta}_1 \dot{\theta}_2)$$

and

$$V_2 = -mg(l_1 \cos\theta_1 + l_2 \cos\theta_2)$$

Thus

$$L_2 = T_2 - V_2$$
$$= \frac{1}{2} m_2 (l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \cos(\theta_1 - \theta_2) \dot{\theta}_1 \dot{\theta}_2) + m_2 g (l_1 \cos\theta_1 + l_2 \cos\theta_2)$$

Add $L_1 + L_2 = L$,

$$L = \frac{1}{2}m_2(l_1^2\dot\theta_1^2 + l_2^2\dot\theta_2^2 + 2l_1l_2\cos(\theta_1-\theta_2)\dot\theta_1\dot\theta_2) + m_2g(l_1\cos\theta_1 + l_2\cos\theta_2) + \frac{1}{2}ml_1\dot\theta_1^2 + m_1gl_1\cos\theta_1$$

Simplify even though it still is pretty messy:

$$L = \frac{1}{2}(m_1+m_2)l_1^2\dot\theta_1^2 + m_2l_1l_2\cos(\theta_1-\theta_2)\dot\theta_1\dot\theta_2 + \frac{1}{2}m_2l_2^2\dot\theta_2^2 + (m_1+m_2)gl_1\cos\theta_1 + m_2gl_2\cos\theta_2$$

This is the Lagrangian for the double pendulum. To find the equations of motion, apply the usual Euler-Lagrangian equations and turn the crank:

For θ_1:

$$\frac{\partial L}{\partial \theta_1} = -m_2l_1l_2\sin(\theta_1-\theta_2)\dot\theta_1\dot\theta_2 - (m_1+m_2)gl_1\sin\theta_1$$

$$\frac{\partial L}{\partial \dot\theta_1} = (m_1+m_2)l_2^2\dot\theta_1 + m_2l_1l_2\cos(\theta_1-\theta_2)\dot\theta_2$$

$$\frac{d}{dt}[\frac{\partial L}{\partial \dot\theta_1}] = (m_1+m_2)l_1^2\ddot\theta_1 + m_2l_1l_2\cos(\theta_1-\theta_2)\ddot\theta_2 + \dot\theta_2\frac{d}{dt}[m_2l_1l_2\cos(\theta_1-\theta_2)]$$

Let's solve this annoying derivative term:

$$\dot\theta_2\frac{d}{dt}[m_2l_1l_2\cos(\theta_1-\theta_2)] = \dot\theta_2m_2l_2l_1\frac{d}{dt}[\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2]$$

Using a trig identity,

$$= \dot\theta_2m_2l_2l_1[-\cos\theta_1\sin\theta_2\dot\theta_2 - \cos\theta_2\sin\theta_1\dot\theta_1 + \sin\theta_1\cos\theta_2\dot\theta_2 + \sin\theta_2\cos\theta_1\dot\theta_1]$$

And then more trig identities to put it back together,

$$= \dot\theta_2m_2l_2l_1[\dot\theta_2\sin(\theta_1-\theta_2) - \dot\theta_1\sin(\theta_1-\theta_2)]$$

$$= \dot\theta_2^2m_2l_2l_1\sin(\theta_1-\theta_2) - m_2l_2l_1\sin(\theta_1-\theta_2)\dot\theta_1\dot\theta_2$$

Plugging this term back into our Euler-Lagrangian formula, the second term of this cancels its positive counterpart:

$$-\frac{\partial L}{\partial \theta} + \frac{d}{dt}[\frac{\partial L}{\partial \dot\theta_1}] = (m_1+m_2)gl_1\sin\theta_1 + (m_1+m_2)l_2^2\ddot\theta_1 + m_2l_1l_2\cos(\theta_1-\theta_2)\ddot\theta_2 + \dot\theta_2^2m_2l_2l_1\sin(\theta_1-\theta_2)$$

Finally, cancel out a l_1 and set to zero for our first equation of motion:

$$(m_1+m_2)g\sin\theta_1 + (m_1+m_2)l_1\ddot\theta_1 + m_2l_2\cos(\theta_1-\theta_2)\ddot\theta_2 + \dot\theta_2^2m_2l_2\sin(\theta_1-\theta_2) = 0$$

Now for θ_2:

$$\frac{\partial L}{\partial \theta_2} = m_2 l_1 l_2 \sin(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2 - m_2 g l_2 \sin\theta_2$$

$$\frac{\partial L}{\partial \dot{\theta}_2} = m_2 l_1 l_2 \cos(\theta_1 - \theta_2)\dot{\theta}_1 + m_2 l_2^2 \dot{\theta}_2$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} = m_2 l_2^2 \ddot{\theta}_2 + \ddot{\theta}_1 m_2 l_2 l_1 \cos(\theta_1 - \theta_2) + \dot{\theta}_1[\frac{d}{dt}(m_2 l_2 l_1 \cos(\theta_1 - \theta_2))]$$

Fortunately this is the same derivative term as before, so we can cut to the chase:

$$= m_2 l_2^2 \ddot{\theta}_2 + \ddot{\theta}_1 m_2 l_2 l_1 \cos(\theta_1 - \theta_2) + m_2 l_1 l_2 \dot{\theta}_1[\dot{\theta}_2 \sin(\theta_1 - \theta_2) - \dot{\theta}_1 \sin(\theta_1 - \theta_2)]$$

Thus

$$-\frac{\partial L}{\partial \theta_2} + \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} = +m_2 g l_2 \sin\theta_2 + m_2 l_2^2 \ddot{\theta}_2 + \ddot{\theta}_1 m_2 l_1 l_2 \cos(\theta_1 - \theta_2) - \dot{\theta}_1^2 m_2 l_1 l_2 \sin(\theta_1 - \theta_2)$$

Cancel out an l_2 this time, set to zero, and we have our second equation of motion:

$$m_2 g \sin\theta_2 + m_2 l_2 \ddot{\theta}_2 + \ddot{\theta}_1 m_2 l_1 \cos(\theta_1 - \theta_2) - \dot{\theta}_1^2 m_2 l_1 \sin(\theta_1 - \theta_2) = 0$$

Both of the equations of motion together along with the Lagrangian:

$$L = \frac{1}{2}(m_1 + m_2) l_1^2 \dot{\theta}_1^2 + m_2 l_1 l_2 \cos(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2 + \frac{1}{2} m_2 l_2^2 \dot{\theta}_2^2 + (m_1 + m_2) g l_1 \cos\theta_1 + m_2 g l_2 \cos\theta_2$$

$$(m_1 + m_2) g \sin\theta_1 + (m_1 + m_2) l_1 \ddot{\theta}_1 + m_2 l_2 \cos(\theta_1 - \theta_2)\ddot{\theta}_2 + \dot{\theta}_2^2 m_2 l_2 \sin(\theta_1 - \theta_2) = 0$$

$$m_2 g \sin\theta_2 + m_2 l_2 \ddot{\theta}_2 + \ddot{\theta}_1 m_2 l_1 \cos(\theta_1 - \theta_2) - \dot{\theta}_1^2 m_2 l_1 \sin(\theta_1 - \theta_2) = 0$$

23. Obtain the equation of motion for a particle falling vertically under the influence of gravity when frictional forces obtainable from a dissipation function $\frac{1}{2}kv^2$ are present. Integrate the equation to obtain the velocity as a function of time and show that the maximum possible velocity for a fall from rest is $v + mg/k$.

Answer:

Work in one dimension, and use the most simple Lagrangian possible:

$$L = \frac{1}{2} m \dot{z}^2 - mgz$$

With dissipation function:

$$F = \frac{1}{2}k\dot{z}^2$$

The Lagrangian formulation is now:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{z}} - \frac{\partial L}{\partial z} + \frac{\partial F}{\partial \dot{z}} = 0$$

Plug and chug and get:

$$m\ddot{z} - mg + k\dot{z} = 0$$

Note that at terminal velocity there is no total force acting on you, gravity matches force due to friction, so $m\ddot{z} = 0$:

$$mg = k\dot{z} \quad \rightarrow \quad \dot{z} = \frac{mg}{k}$$

But lets integrate like the problem asks. Let $f = \dot{z} - \frac{mg}{k}$ and substitute into the equation of motion:

$$m\ddot{z} - mg + k\dot{z} = 0$$
$$\frac{m\ddot{z}}{k} - \frac{mg}{k} + \dot{z} = 0$$
$$\frac{m\ddot{z}}{k} + f = 0$$

Note that $f' = \ddot{z}$. Thus

$$\frac{mf'}{k} + f = 0$$
$$\frac{f'}{f} = -\frac{k}{m}$$
$$\ln f = -\frac{k}{m}t + C$$
$$f = Ce^{-\frac{k}{m}t}$$

Therefore

$$\dot{z} - \frac{mg}{k} = Ce^{-\frac{k}{m}t}$$

Plugging in the boundary conditions, that at $t = 0$, $\dot{z} = 0$, we solve for C

$$-\frac{mg}{k} = C$$

Thus

$$\dot{z} - \frac{mg}{k} = -\frac{mg}{k}e^{-\frac{k}{m}t}$$

and with $t \to \infty$ we have finally

$$\dot{z} = \frac{mg}{k}$$

24. A spring of rest length L_a (no tension) is connected to a support at one end and has a mass M attached at the other. Neglect the mass of the spring, the dimension of the mass M, and assume that the motion is confined to a vertical plane. Also, assume that the spring only stretches without bending but it can swing in the plane.

1. Using the angular displacement of the mass from the vertical and the length that the string has stretched from its rest length (hanging with the mass m), find Lagrange's equations.
2. Solve these equations fro small stretching and angular displacements.
3. Solve the equations in part (1) to the next order in both stretching and angular displacement. This part is amenable to hand calculations. Using some reasonable assumptions about the spring constant, the mass, and the rest length, discuss the motion. Is a resonance likely under the assumptions stated in the problem?
4. (For analytic computer programs.) Consider the spring to have a total mass $m << M$. Neglecting the bending of the spring, set up Lagrange's equations correctly to first order in m and the angular and linear displacements.
5. (For numerical computer analysis.) Make sets of reasonable assumptions of the constants in part (1) and make a single plot of the two coordinates as functions of time.

Answer:

This is a spring-pendulum. It's kinetic energy is due to translation only.

$$T = \frac{1}{2}m(\dot{r}^2 + (r\dot{\theta})^2)$$

The more general form of v is derived in problem 15 if this step was not clear. Just disregard ϕ direction. Here r signifies the total length of the spring, from support to mass at any time.

As in problem 22, the potential has a term dependent on gravity, but it also has the potential of your normal spring.

$$V = -mgr\cos\theta + \frac{1}{2}k(r - L_a)^2$$

Note that the potential due to gravity depends on the total length of the spring, while the potential due to the spring is only dependent on the stretching from its natural length. Solving for the Lagrangian:

$$L = T - V = \frac{1}{2}m(\dot{r}^2 + (r\dot{\theta})^2) + mgr\cos\theta - \frac{1}{2}k(r - L_a)^2$$

Lets solve for Lagrange's equations now.

For r:

$$\frac{\partial L}{\partial r} = mg\cos\theta + mr\dot{\theta}^2 - k(r - L_a)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = m\ddot{r}$$

For θ:

$$\frac{\partial L}{\partial \theta} = -mgr\sin\theta$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{d}{dt}(mr^2\dot{\theta}) = mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta}$$

Bring all the pieces together to form the equations of motion:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = m\ddot{r} - mr\dot{\theta}^2 + k(r - L_a) - mg\cos\theta = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} + mgr\sin\theta = 0$$

For part b, we are to solve these equations for small stretching and angular displacements. Simplify the equations above by canceling out m's, r's and substituting θ for $\sin\theta$, and 1 for $\cos\theta$.

$$\ddot{r} - r\dot{\theta}^2 + \frac{k}{m}(r - L_a) - g = 0$$

$$\ddot{\theta} + \frac{2\dot{r}}{r}\dot{\theta} + \frac{g}{r}\theta = 0$$

Solve the first equation, for r, with the initial condition that $\theta_0 = 0$, $\dot{\theta}_0 = 0$, $r_0 = 0$ and $\dot{r}_0 = 0$:

$$r = L_a + \frac{mg}{k}$$

Solve the second equation, for θ, with the same initial conditions:

$$\theta = 0$$

This is the solution of the Lagrangian equations that make the generalized force identically zero. To solve the next order, change variables to measure deviation from equilibrium.

$$x = r - (L_a + \frac{mg}{k}), \quad \theta$$

Substitute the variables, keep only terms to 1st order in x and θ and the solution is:

$$\ddot{x} = -\frac{k}{m}x \quad \ddot{\theta} = -\frac{g}{L_a + \frac{m}{k}g}\theta$$

In terms of the original coordinates r and θ, the solutions to these are:

$$r = L_a + \frac{mg}{k} + A\cos(\sqrt{\frac{k}{m}}t + \phi)$$

$$\theta = B\cos(\sqrt{\frac{kg}{kL_a + mg}}t + \phi')$$

The phase angles, ϕ and ϕ', and amplitudes A and B are constants of integration and fixed by the initial conditions. Resonance is very unlikely with this system. The spring pendulum is known for its nonlinearity and studies in chaos theory.

Homework 1: # 1.21, 2.7, 2.12

Michael Good

Sept 3, 2004

> 1.21. Two mass points of mass m_1 and m_2 are connected by a string passing through a hole in a smooth table so that m_1 rests on the table surface and m_2 hangs suspended. Assuming m_2 moves only in a vertical line, what are the generalized coordinates for the system? Write the Lagrange equations for the system and, if possible, discuss the physical significance any of them might have. Reduce the problem to a single second-order differential equation and obtain a first integral of the equation. What is its physical significance? (Consider the motion only until m_1 reaches the hole.)

Answer:

The generalized coordinates for the system are θ, the angle m_1 moves round on the table, and r the length of the string from the hole to m_1. The whole motion of the system can be described by just these coordinates. To write the Lagrangian, we will want the kinetic and potential energies.

$$T = \frac{1}{2}m_2\dot{r}^2 + \frac{1}{2}m_1(\dot{r}^2 + r^2\dot{\theta}^2)$$

$$V = -m_2 g(R - r)$$

The kinetic energy is just the addition of both masses, while V is obtained so that $V = -mgR$ when $r = 0$ and so that $V = 0$ when $r = R$.

$$L = T - V = \frac{1}{2}(m_2 + m_1)\dot{r}^2 + \frac{1}{2}m_1 r^2\dot{\theta}^2 + m_2 g(R - r)$$

To find the Lagrangian equations or equations of motion, solve for each component:

$$\frac{\partial L}{\partial \theta} = 0$$

$$\frac{\partial L}{\partial \dot{\theta}} = m_1 r^2 \dot{\theta}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{d}{dt}(m_1 r^2\dot{\theta}) = m_1 r^2\ddot{\theta} + 2m_1 r\dot{r}\dot{\theta}$$

Thus

$$m_1 r(r\ddot{\theta} + 2\dot{\theta}\dot{r}) = 0$$

and

$$\frac{\partial L}{\partial r} = -m_2 g + m_1 r\dot{\theta}^2$$
$$\frac{\partial L}{\partial \dot{r}} = (m_2 + m_1)\dot{r}$$
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = (m_2 + m_1)\ddot{r}$$

Thus

$$m_2 g - m_1 r\dot{\theta}^2 + (m_2 + m_1)\ddot{r} = 0$$

Therefore our equations of motion are:

$$\frac{d}{dt}(m_1 r^2\dot{\theta}) = m_1 r(r\ddot{\theta} + 2\dot{\theta}\dot{r}) = 0$$

$$m_2 g - m_1 r\dot{\theta}^2 + (m_2 + m_1)\ddot{r} = 0$$

See that $m_1 r^2\dot{\theta}$ is constant, because $\frac{d}{dt}(m_1 r^2\dot{\theta}) = 0$. It is angular momentum. Now the Lagrangian can be put in terms of angular momentum and will lend the problem to interpretation. We have $\dot{\theta} = l/m_1 r^2$, where l is angular momentum. The equation of motion

$$m_2 g - m_1 r\dot{\theta}^2 + (m_2 + m_1)\ddot{r} = 0$$

becomes

$$(m_1 + m_2)\ddot{r} - \frac{l^2}{m_1 r^3} + m_2 g = 0$$

The problem has been reduced to a single non-linear second-order differential equation. The next step is a nice one to notice. If you take the first integral you get

$$\frac{1}{2}(m_1 + m_2)\dot{r}^2 + \frac{l^2}{2m_1 r^2} + m_2 g r + C = 0$$

To see this, check by assuming that $C = -m_2 g R$:

$$\frac{d}{dt}(\frac{1}{2}(m_1 + m_2)\dot{r}^2 + \frac{l^2}{2m_1 r^2} - m_2 g(R - r)) = (m_1 + m_2)\dot{r}\ddot{r} - \frac{l^2}{m_1 r^3}\dot{r} + m_2 g\dot{r} = 0$$

$$(m_1 + m_2)\ddot{r} - \frac{l^2}{m_1 r^3} + m_2 g = 0$$

Because this term is T *plus* V, this is the total energy, and because its time derivative is constant, energy is conserved.

2.7 In Example 2 of Section 2.1 we considered the problem of the minimum surface of revolution. Examine the symmetric case $x_1 = x_2$, $y_2 = -y_1 > 0$, and express the condition for the parameter a as a transcendental equation in terms of the dimensionless quantities $k = x_2/a$, and $\alpha = y_2/x_2$. Show that for α greater than a certain value α_0 two values of k are possible, for $\alpha = \alpha_0$ only one value of k is possible, while if $\alpha < \alpha_0$ no real value of k (or a) can be found, so that no catenary solution exists in this region. Find the value of α_0, numerically if necessary.

Answer:

Starting with Goldstein's form for a catenary, in section 2.2, not 2.1,

$$x = a \cosh \frac{y - b}{a}$$

and recognizing by symmetry that the soap film problem and the catenary problem are the same. In Marion and Thorton this is made clear (pg 222). Also, in a similar way to MathWorld's analysis of a surface of revolution, it is clear that y and x, when interchanged, change the shape of the catenary to be about the x-axis.

$$y = a \cosh \frac{x - b}{a}$$

To preserve symmetry, $x_1 = -x_2$ and $y_2 = y_1$. This switch makes sense because if you hang a rope from two points, its going to hang between the points with a droopy curve, and fall straight down after the points. This shaped revolved around the x-axis looks like a horizontal worm hole. This is the classic catenary curve, or catenoid shape. The two shapes are physically equivalent, and take on different mathematical forms. With this, we see that

$$y_1 = a \cosh \frac{x_1 - b}{a} \qquad y_2 = a \cosh \frac{x_2 - b}{a}$$

holds. The two endpoints are (x_0, y_0) and $(-x_0, y_0)$.

Thus

$$y_0 = a \cosh \frac{x_0 - b}{a} = a \cosh \frac{-x_0 - b}{a}$$

and because

$$\cosh(-x) = \cosh(x)$$

we have

$$\cosh \frac{-x_0 + b}{a} = \cosh \frac{-x_0 - b}{a}$$

$$-x_0 + b = -x_0 - b$$

$$b = -b \qquad b = 0$$

By symmetry, with the center of the shape or rings at the origin, $b = 0$, simplifies the problem to a much nicer form:

$$y = a \cosh \frac{x}{a}$$

Including our end points:

$$y_0 = a \cosh \frac{x_0}{a}$$

In terms of the dimensionless quantities,

$$\rho = x_2/a \qquad \beta = \frac{1}{\alpha} = y_2/x_2$$

,

the equation is

$$y_0 = a \cosh \frac{x_0}{a}$$

$$\frac{y_0}{x_0} = \frac{a \cosh \rho}{x_0}$$

$$\frac{y_0}{x_0} = \frac{\cosh(\rho)}{\rho}$$

$$\beta = \frac{\cosh(\rho)}{\rho}$$

The minimum value of β in terms of ρ can be found by taking the derivative, and setting to zero:

$$0 = \frac{d}{d\rho}(\frac{1}{\rho}\cosh\rho) = \frac{1}{\rho}\sinh\rho - \frac{\cosh\rho}{\rho^2}$$

$$\sinh\rho = \frac{\cosh\rho}{\rho}$$

$$\rho = \coth\rho$$

Thus, solved numerically, $\rho \approx 1.2$. Plugging this in to find β_0, the value is:

$$\beta_0 \approx 1.51$$

Since

$$\beta_0 = \frac{1}{\alpha_0}$$

,

$$\alpha_0 \approx .66$$

This symmetric but physically equivalent example is not what the problem asked for, but I think its interesting. If we start at Goldstein's equation, again, only this time recognize $b = 0$ due to symmetry from the start, the solution actually follows more quickly.

$$x = a \cosh \frac{y}{a}$$

$$\frac{x}{a} = \cosh \frac{x}{a}\frac{y}{x}$$

Using, the dimensional quantities defined in the problem,

$$k = \frac{x_2}{a} \qquad \alpha = \frac{y_2}{x_2}$$

we have

$$k = \cosh k\alpha$$

Taking the derivative with respect to k,

$$1 = \alpha_0 \sinh k\alpha_0$$

Using the hyperbolic identity,

$$\cosh^2 A - \sinh^2 A = 1$$

a more manageable expression in terms of k and α becomes apparent,

$$k^2 - \frac{1}{\alpha_0^2} = 1$$

$$\alpha_0 = \frac{1}{\sqrt{k^2 - 1}}$$

Plug this into $k = \cosh k\alpha$

$$k = \cosh \frac{k}{\sqrt{k^2 - 1}}$$

Solving this numerically for k yields,

$$k \approx 1.81$$

Since

$$\alpha_0 = \frac{1}{\sqrt{k^2 - 1}} \quad \Rightarrow \quad \alpha_0 \approx .66$$

If $\alpha < \alpha_0$, two values of k are possible. If $\alpha > \alpha_0$, no real values of k exist, but if $\alpha = \alpha_0$ then only $k \approx 1.81$ will work. This graph is $arccosh(k)/k = \alpha$ and looks like a little hill. It can be graphed by typing acosh(x)/x on a free applet at http://www.tacoma.ctc.edu/home/jkim/gcalc.html.

2.12 The term *generalized mechanics* has come to designate a variety of classical mechanics in which the Lagrangian contains time derivatives of q_i higher than the first. Problems for which triple dot $x = f(x, \dot{x}, \ddot{x}, t)$ have been referred to as 'jerky' mechanics. Such equations of motion have interesting applications in chaos theory (cf. Chapter 11). By applying the mehtods of the calculus of variations, show that if there is a Lagrangian of the form $L(q_i, \dot{q}_i\ddot{q}_i, t)$, and Hamilton's principle holds with the zero variation of both q_i and $\dot{q}_i$ at the end points, then the corresponding Euler-Lagrange equations are

$$\frac{d^2}{dt^2}(\frac{\partial L}{\partial \ddot{q}_i}) - \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i}) + \frac{\partial L}{\partial q_i} = 0, \qquad i = 1, 2, ..., n.$$

Apply this result to the Lagrangian

$$L = -\frac{m}{2}q\ddot{q} - \frac{k}{2}q^2$$

Do you recognize the equations of motion?

Answer:

If there is a Lagrangian of the form

$$L = L(q_i, \dot{q}_i, \ddot{q}_i, t)$$

and Hamilton's principle holds with the zero variation of both q_i and $\dot{q}_i$ at the end points, then we have:

$$I = \int_1^2 L(q_i, \dot{q}_i, \ddot{q}_i, t)dt$$

and

$$\frac{\partial I}{\partial \alpha}d\alpha = \int_1^2 \sum_i (\frac{\partial L}{\partial q_i}\frac{\partial q_i}{\partial \alpha_i}d\alpha + \frac{\partial L}{\partial \dot{q}_i}\frac{\partial \dot{q}_i}{\partial \alpha_i}d\alpha + \frac{\partial L}{\partial \ddot{q}_i}\frac{\partial \ddot{q}_i}{\partial \alpha_i}d\alpha)dt$$

To make life easier, we're going to assume the Einstein summation convention, as well as drop the indexes entirely. In analogy with the differential quantity, Goldstein Equation (2.12), we have

$$\delta q = \frac{\partial q}{\partial \alpha}d\alpha$$

Applying this we have

$$\delta I = \int_1^2 (\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\frac{\partial \dot{q}}{\partial \alpha}d\alpha + \frac{\partial L}{\partial \ddot{q}}\frac{\partial \ddot{q}}{\partial \alpha}d\alpha)dt$$

The indexes are invisible and the two far terms are begging for some mathematical manipulation. Integration by parts on the middle term yields, in analogy to Goldstein page 44,

$$\int_1^2 \frac{\partial L}{\partial \dot{q}} \frac{\partial^2 q}{\partial \alpha \partial t} dt = \frac{\partial L}{\partial \dot{q}} \frac{\partial q}{\partial \alpha}\bigg|_1^2 - \int_1^2 \frac{\partial q}{\partial \alpha} \frac{d}{dt} (\frac{\partial L}{\partial \dot{q}}) dt$$

This first term on the right is zero because the condition exists that all the varied curves pass through the fixed end points and thus the partial derivative of q wrt to α at x_1 and x_2 vanish. Substituting back in, we have:

$$\delta I = \int_1^2 (\frac{\partial L}{\partial q} \delta q - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \delta q + \frac{\partial L}{\partial \ddot{q}} \frac{\partial \ddot{q}}{\partial \alpha} d\alpha) dt$$

Where we used the definition $\delta q = \frac{\partial q}{\partial \alpha} d\alpha$ again. Now the last term needs attention. This requires integration by parts twice. Here goes:

$$\int_1^2 \frac{\partial L}{\partial \ddot{q}} \frac{\partial \ddot{q}}{\partial \alpha} dt = \frac{\partial L}{\partial \ddot{q}} \frac{\partial^2 q}{\partial t \partial \alpha}\bigg|_1^2 - \int_1^2 \frac{\partial^2 q}{\partial t \partial \alpha} \frac{d}{dt} (\frac{\partial L}{\partial \ddot{q}}) dt$$

Where we used $\int v du = uv - \int v du$ as before. The first term vanishes once again, and we are still left with another integration by parts problem. Turn the crank again.

$$-\int_1^2 \frac{\partial^2 q}{\partial t \partial \alpha} \frac{d}{dt} (\frac{\partial L}{\partial \ddot{q}}) dt = \frac{d}{dt} \frac{\partial L}{\partial \ddot{q}} \frac{\partial q}{\partial \alpha}\bigg|_1^2 - \int_1^2 -\frac{\partial q}{\partial \alpha} \frac{d^2}{dt^2} \frac{\partial L}{\partial \ddot{q}} dt$$

First term vanishes for the third time, and we have

$$\int_1^2 \frac{\partial L}{\partial \ddot{q}} \frac{\partial \ddot{q}}{\partial \alpha} dt = \int_1^2 \frac{\partial q}{\partial \alpha} \frac{d^2}{dt^2} \frac{\partial L}{\partial \ddot{q}} dt$$

Plugging back in finally, and using the definition of our δq, we get closer

$$\delta I = \int_1^2 (\frac{\partial L}{\partial q} \delta q - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \delta q + \frac{d^2}{dt^2} \frac{\partial L}{\partial \ddot{q}} \delta q) dt$$

Gathering δq's, throwing our summation sign and index's back in, and applying Hamiliton's principle:

$$\delta I = \int_1^2 \sum_i (\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} + \frac{d^2}{dt^2} \frac{\partial L}{\partial \ddot{q}_i}) \delta q_i dt = 0$$

We know that since q variables are independent, the variations δq_i are independent and we can apply the calculus of variations lemma, (Goldstein, Eq. 2.10) and see that $\delta I = 0$ requires that the coefficients of δq_i separately vanish, one by one:

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} + \frac{d^2}{dt^2} \frac{\partial L}{\partial \ddot{q}_i} = 0 \qquad i = 1, 2, ...n.$$

Applying this result to the Lagrangian,

$$L = -\frac{1}{2}mq\ddot{q} - \frac{k}{2}q^2$$

yields

$$\frac{\partial L}{\partial q} = -\frac{1}{2}m\ddot{q} - kq$$

$$-\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0$$

$$\frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{q}} = \frac{d}{dt}(\frac{d}{dt}(-\frac{1}{2}mq)) = \frac{d}{dt}(-\frac{1}{2}m\dot{q}) = -\frac{1}{2}m\ddot{q}$$

Adding them up:

$$-m\ddot{q} - kq = 0$$

This is interesting because this equation of motion is just Hooke's Law. This crazy looking Lagrangian yields the same equation for simple harmonic motion using the 'jerky' form of Lagrangian's equations. It's interesting to notice that if the familiar Lagrangian for a simple harmonic oscillator (SHO) plus an extra term is used, the original Lagrangian can be obtained.

$$L = L_{SHO} + \frac{d}{dt}(-\frac{mq\dot{q}}{2})$$

$$L = \frac{m\dot{q}^2}{2} - \frac{kq^2}{2} + \frac{d}{dt}(-\frac{mq\dot{q}}{2})$$

$$L = \frac{m\dot{q}^2}{2} - \frac{kq^2}{2} - \frac{mq\ddot{q}}{2} - \frac{m\dot{q}^2}{2}$$

$$L = -\frac{mq\ddot{q}}{2} - \frac{kq^2}{2}$$

This extra term, $\frac{d}{dt}(-\frac{mq\dot{q}}{2})$ probably represents constraint. The generalized force of constraint is the Lagrange multipliers term that is added to the original form of Lagrange's equations.

Homework 8: # 5.4, 5.6, 5.7, 5.26

Michael Good

Oct 21, 2004

> 5.4
> Derive Euler's equations of motion, Eq. (5.39'), from the Lagrange equation of motion, in the form of Eq. (1.53), for the generalized coordinate ψ.

Answer:

Euler's equations of motion for a rigid body are:

$$I_1\dot{\omega}_1 - \omega_2\omega_3(I_2 - I_3) = N_1$$
$$I_2\dot{\omega}_2 - \omega_3\omega_1(I_3 - I_1) = N_2$$
$$I_3\dot{\omega}_3 - \omega_1\omega_2(I_1 - I_2) = N_3$$

The Lagrangian equation of motion is in the form

$$\frac{d}{dt}(\frac{\partial T}{\partial \dot{q}_j}) - \frac{\partial T}{\partial q_j} = Q_j$$

The kinetic energy for rotational motion is

$$T = \sum_i^3 \frac{1}{2} I_i \omega_i^2$$

The components of the angular velocity in terms of Euler angles for the body set of axes are

$$\omega_1 = \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi$$
$$\omega_2 = \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi$$
$$\omega_3 = \dot{\phi}\cos\theta + \dot{\psi}$$

Solving for the equation of motion using the generalized coordinate ψ:

$$\frac{d}{dt}(\frac{\partial T}{\partial \dot{\psi}}) - \frac{\partial T}{\partial \psi} = N_\psi$$

$$\sum_i^3 I_i\omega_i \frac{\partial \omega_i}{\partial \psi} - \frac{d}{dt}\sum_i^3 I_i\omega_i \frac{\partial \omega_i}{\partial \dot{\psi}} = N_\psi$$

Now is a good time to pause and calculate the partials of the angular velocities,

$$\frac{\partial \omega_1}{\partial \psi} = -\dot{\theta} \sin \psi + \dot{\phi} \sin \theta \cos \psi$$

$$\frac{\partial \omega_2}{\partial \psi} = -\dot{\theta} \cos \psi - \dot{\phi} \sin \theta \sin \psi$$

$$\frac{\partial \omega_3}{\partial \psi} = 0$$

and

$$\frac{\partial \omega_1}{\partial \dot{\psi}} = \frac{\partial \omega_2}{\partial \dot{\psi}} = 0$$

$$\frac{\partial \omega_3}{\partial \dot{\psi}} = 1$$

Now we have all the pieces of the puzzle, explicitly

$$\sum_i^3 I_i \omega_i \frac{\partial \omega_i}{\partial \psi} - \frac{d}{dt} \sum_i^3 I_i \omega_i \frac{\partial \omega_i}{\partial \dot{\psi}} = N_\psi$$

$$I_1 \omega_1 (-\dot{\theta} \sin \psi + \dot{\phi} \sin \theta \cos \psi) + I_2 \omega_2 (-\dot{\theta} \cos \psi - \dot{\phi} \sin \theta \sin \psi) - \frac{d}{dt} I_3 \omega_3 = N_\psi$$

This is, pulling out the negative sign on the second term,

$$I_1 \omega_1 (\omega_2) - I_2 \omega_2 (\omega_1) - I_3 \dot{\omega}_3 = N_\psi$$

$$I_3 \dot{\omega}_3 - \omega_1 \omega_2 (I_1 - I_2) = N_\psi$$

And through cyclic permutations

$$I_2 \dot{\omega}_2 - \omega_3 \omega_1 (I_3 - I_1) = N_2$$

$$I_1 \dot{\omega}_1 - \omega_2 \omega_3 (I_2 - I_3) = N_1$$

we have the rest of Euler's equations of motion for a rigid body.

5.6

- Show that the angular momentum of the torque-free symmetrical top rotates in the body coordinates about the symmetry axis with an angular frequency ω. Show also that the symmetry axis rotates in space about the fixed direction of the angular momentum with angular frequency

$$\dot{\phi} = \frac{I_3\omega_3}{I_1 \cos\theta}$$

where ϕ is the Euler angle of the line of nodes with respect to the angular momentum as the space z axis.

- Using the results of Exercise 15, Chapter 4, show that ω rotates in space about the angular momentum with the same frequency $\dot{\phi}$, but that the angle θ' between ω and L is given by

$$\sin\theta' = \frac{\Omega}{\dot{\phi}} \sin\theta''$$

where θ'' is the inclination of ω to the symmetry axis. Using the data given in Section 5.6, show therefore that Earth's rotation axis and axis of angular momentum are never more than 1.5 cm apart on Earth's surface.

- Show from parts (a) and (b) that the motion of the force-free symmetrical top can be described in terms of the rotation of a cone fixed in the body whose axis is the symmetry axis, rolling on a fixed cone in space whose axis is along the angular momentum. The angular velocity vector is along the line of contact of the two cones. Show that the same description follows immediately from the Poinsot construction in terms of the inertia ellipsoid.

Answer:

Marion shows that the angular momentum of the torque-free symmetrical top rotates in the body coordinates about the symmetry axis with an angular frequency ω more explicitly than Goldstein. Beginning with Euler's equation for force-free, symmetric, rigid body motion, we see that $\omega_3 = \textit{constant}$. The other Euler equations are

$$\dot{\omega}_1 = -(\frac{I_3 - I}{I}\omega_3)\omega_2$$

$$\dot{\omega}_2 = -(\frac{I_3 - I}{I}\omega_3)\omega_1$$

Solving these, and by already making the substitution, because we are dealing with constants,

$$\Omega = \frac{I_3 - I}{I}\omega_3$$

we get

$$(\dot{\omega}_1 + i\dot{\omega}_2) - i\Omega(\omega_1 + i\omega_2) = 0$$

Let

$$q = \omega_1 + i\omega_2$$

Now

$$\dot{q} - i\Omega q = 0$$

has solution

$$q(t) = Ae^{i\Omega t}$$

this is

$$\omega_1 + i\omega_2 = A\cos\Omega t + iA\sin\Omega t$$

and we see

$$\omega_1(t) = A\cos\Omega t$$

$$\omega_2(t) = A\sin\Omega t$$

The x_3 axis is the symmetry axis of the body, so the angular velocity vector precesses about the body x_3 axis with a constant angular frequency

$$\Omega = \frac{I_3 - I}{I}\omega_3$$

.

$$\dot{\phi} = \frac{I_3\omega_3}{I_1\cos\theta}$$

To prove

$$\dot{\phi} = \frac{I_3\omega_3}{I_1\cos\theta}$$

We may look at the two cone figure angular momentum components, where L is directed along the vertical space axis and θ is the angle between the space and body vertical axis.

$$L_1 = 0$$

$$L_2 = L\sin\theta$$

$$L_3 = L\cos\theta$$

If α is the angle between ω and the vertical body axis, then

$$\omega_1 = 0$$

$$\omega_2 = \omega \sin\alpha$$

$$\omega_3 = \omega \cos\alpha$$

The angular momentum components in terms of α may be found

$$L_1 = I_1\omega_1 = 0$$

$$L_2 = I_1\omega_2 = I_1\omega\sin\alpha$$

$$L_3 = I_3\omega_3 = I_3\omega\cos\alpha$$

Using the Euler angles in the body frame, we may find, (using the instant in time where x_2 is in the plane of x_3, ω, and L, where $\psi = 0$) ,

$$\omega_2 = \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi$$

$$\omega_2 = \dot{\phi}\sin\theta$$

This is

$$\dot{\phi} = \frac{\omega_2}{\sin\theta} = \frac{\omega\sin\alpha}{\sin\theta} = \omega(\frac{L_2}{I_1\omega})\frac{L}{L_2} = \frac{L}{I_1}$$

Plugging in L_3

$$\dot{\phi} = \frac{L}{I_1} = \frac{L_3}{I_1\cos\theta} = \frac{I_3\omega_3}{I_1\cos\theta}$$

A simple way to show

$$\sin\theta' = \frac{\Omega}{\dot{\phi}}\sin\theta''$$

may be constructed by using the cross product of $\omega \times L$ and $\omega \times x_3$.

$$|\omega \times L| = \omega L\sin\theta' = L\sqrt{\omega_x^2 + \omega_y^2}$$

Using the angular velocity components in terms of Euler angles in the space fixed frame, this is equal to

$$\omega L\sin\theta' = L\dot{\phi}\sin\theta$$

with θ fixed, and $\dot{\theta} = 0$. For $\omega \times x_3$ we have

$$|\omega \times x_3| = \omega\sin\theta'' = \sqrt{\omega_{x'}^2 + \omega_{y'}^2}$$

Using the angular velocity components in terms of Euler angles in the body fixed frame, this is equal to

$$\omega \sin\theta'' = \dot{\phi}\sin\theta$$

Using these two expressions, we may find their ratio

$$\frac{\omega L \sin\theta'}{\omega \sin\theta''} = \frac{L\dot{\phi}\sin\theta}{\dot{\phi}\sin\theta}$$

$$\frac{\sin\theta'}{\sin\theta''} = \frac{\dot{\psi}}{\dot{\phi}}$$

Because $\dot{\psi} = \Omega$

$$\sin\theta' = \frac{\Omega}{\dot{\phi}}\sin\theta''$$

To show that the Earth's rotation axis and axis of angular momentum are never more than 1.5cm apart on the Earth's surface, the following approximations may be made, $\sin\theta' \approx \theta'$, $\cos\theta \approx 1$, $\sin\theta'' \approx \theta''$, and $I_1/I_3 \approx 1$. Earth is considered an oblate spheroid, $I_3 > I_1$ and the data says there is $10m$ for amplitude of separation of pole from rotation axis. Using

$$\sin\theta' = \frac{\Omega}{\dot{\phi}}\sin\theta''$$

$$\dot{\phi} = \frac{I_3\omega_3}{I_1\cos\theta}$$

$$\Omega = \frac{I_3 - I_1}{I_1}\omega_3$$

we have

$$\sin\theta' = \frac{I_3 - I_1}{I_1}\omega_3\frac{I_1\cos\theta}{I_3\omega_3}\sin\theta''$$

Applying the approximations

$$\theta' = \frac{I_3 - I_1}{I_1}\theta''$$

$$\theta' = \frac{d}{R} = \frac{I_3 - I_1}{I_1}\frac{s}{R}$$

where R is the radius of the Earth, and s is the average distance of separation, which we will assume is half the amplitude, $5\,m$.

$$d = \frac{I_3 - I_1}{I_1}s = (.00327)(5) = 1.6\,cm$$

Force free motion means the angular momentum vector L is constant in time and stationary, as well as the rotational kinetic energy. (because the center of mass of the body is fixed). So because $T = \frac{1}{2}\omega \cdot L$ is constant, ω precesses around with a constant angle. This tracing is called the space cone, only if L is lined up with x_3 space axis. Proving that L, x_3 and ω all lie in the same plane will show that this space cone is traced out by ω. This results from $I_1 = I_2$ as shown below:

$$L \cdot (\omega \times e_3) = 0$$

because

$$\omega \times e_3 = \omega_2 e_1 - \omega_1 e_2$$

$$L \cdot (\omega \times e_3) = I_1\omega_1\omega_2 - I_2\omega_1\omega_2 = 0$$

Because $I_1 = I_2$.

Now the symmetry axis of the body has the angular velocity ω precessing around it with a constant angular frequency Ω. Thus another cone is traced out, the body cone. So we have two cones, hugging each other with ω in the direction of the line of contact.

5.7
For the general asymmetrical rigid body, verify analytically the stability theorem shown geometrically above on p. 204 by examining the solution of Euler's equations for small deviations from rotation about each of the principal axes. The direction of ω is assumed to differ so slightly from a principal axis that the component of ω along the axis can be taken as constant, while the product of components perpendicular to the axis can be neglected. Discuss the boundedness of the resultant motion for each of the three principal axes.

Answer:

Marion and Thornton give a clear analysis of the stability of a general rigid body. First lets define our object to have distinct principal moments of inertia. $I_1 < I_2 < I_3$. Lets examine the x_1 axis first. We have $\omega = \omega_1 e_1$ if we spin it around the x_1 axis. Apply some small perturbation and we get

$$\omega = \omega_1 e_1 + k e_2 + p e_3$$

In the problem, we are told to neglect the product of components perpendicular to the axis of rotation. This is because k and p are so small. The Euler equations

$$I_1\dot{\omega}_1 - \omega_2\omega_3(I_2 - I_3) = 0$$
$$I_2\dot{\omega}_2 - \omega_3\omega_1(I_3 - I_1) = 0$$
$$I_3\dot{\omega}_3 - \omega_1\omega_2(I_1 - I_2) = 0$$

become

$$I_1\dot{\omega}_1 - kp(I_2 - I_3) = 0$$
$$I_2\dot{k} - p\omega_1(I_3 - I_1) = 0$$
$$I_3\dot{p} - \omega_1 k(I_1 - I_2) = 0$$

Neglecting the product $pk \approx 0$, we see ω_1 is constant from the first equation. Solving the other two yields

$$\dot{k} = (\frac{I_3 - I_1}{I_2}\omega_1)p$$
$$\dot{p} = (\frac{I_1 - I_2}{I_3}\omega_1)k$$

To solve we may differentiate the first equation, and plug into the second:

$$\ddot{k} = (\frac{I_3 - I_1}{I_2}\omega_1)\dot{p} \quad \rightarrow \quad \ddot{k} + (\frac{(I_1 - I_3)(I_1 - I_2)}{I_2 I_3}\omega_1^2)k = 0$$

Solve for $k(t)$:

$$k(t) = Ae^{i\Omega_{1k}t} + Be^{-i\Omega_{1k}t}$$

with

$$\Omega_{1k} = \omega_1\sqrt{\frac{(I_1 - I_3)(I_1 - I_2)}{I_2 I_3}}$$

Do this for $p(t)$ and you get

$$\Omega_{1k} = \Omega_{1p} \equiv \Omega_1$$

Cyclic permutation for the other axes yields

$$\Omega_1 = \omega_1\sqrt{\frac{(I_1 - I_3)(I_1 - I_2)}{I_2 I_3}}$$
$$\Omega_2 = \omega_2\sqrt{\frac{(I_2 - I_1)(I_2 - I_3)}{I_3 I_1}}$$
$$\Omega_3 = \omega_3\sqrt{\frac{(I_3 - I_2)(I_3 - I_1)}{I_1 I_2}}$$

Note that the only unstable motion is about the x_2 axis, because $I_2 < I_3$ and we obtain a negative sign under the square root, Ω_2 is imaginary and the perturbation increases forever with time. Around the x_2 axis we have unbounded motion. Thus we conclude that only the largest and smallest moment of inertia rotations are stable, and the intermediate principal axis of rotation is unstable.

5.26
For the axially symmetric body precessing uniformly in the absence of torques, find the analytical solutions for the Euler angles as a function of time.

Answer:

For an axially symmetric body, symmetry axis L_z, we have $I_1 = I_2$, and Euler's equations are

$$I_1\dot{\omega}_1 = (I_1 - I_3)\omega_2\omega_3$$
$$I_2\dot{\omega}_2 = (I_3 - I_1)\omega_1\omega_3$$
$$I_3\dot{\omega}_3 = 0$$

This is equation (5.47) of Goldstein, only without the typos. Following Goldstein,

$$\omega_1 = A\cos\Omega t$$
$$\omega_2 = A\sin\Omega t$$

where

$$\Omega = \frac{I_3 - I_1}{I_1}\omega_3$$

Using the Euler angles in the body fixed frame,

$$\omega_1 = \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi$$
$$\omega_2 = \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi$$
$$\omega_3 = \dot{\phi}\cos\theta + \dot{\psi}$$

we have

$$\omega_1 = \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi = A\sin(\Omega t + \delta) \quad (1)$$
$$\omega_2 = \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi = A\cos(\Omega t + \delta) \quad (2)$$
$$\omega_3 = \dot{\phi}\cos\theta + \dot{\psi} = constant \quad (3)$$

Multiplying the left hand side of (1) by $\cos\psi$ and the left hand side of (2) by $\sin\psi$, and subtracting them yields

$$[\dot{\phi}\sin\theta\sin\psi\cos\psi + \dot{\theta}\cos^2\psi] - [\dot{\phi}\sin\theta\cos\psi\sin\psi - \dot{\theta}\sin^2\psi] = \dot{\theta}$$

Thus we have

$$\dot{\theta} = A\sin(\Omega t + \delta)\cos\psi + A\cos(\Omega t + \delta)\sin\psi$$
$$\dot{\theta} = A\sin(\Omega t + \delta + \psi)$$

I assume uniform precession means $\dot{\theta} = 0$, no nutation or bobbing up and down. Thus

$$\Omega t + \delta + \psi = n\pi$$

with $n = 0, \pm 1, \pm 2...$, if $n = 0$

$$\psi = -\Omega t + \psi_0$$

where ψ_0 is the initial angle from the $x - axis$. From this, $\dot{\psi} = -\Omega$.

If we multiply the left hand side of (1) by $\sin\psi$ and the left hand side of (2) by $\cos\psi$, and add them:

$$[\dot{\phi}\sin\theta\sin^2\psi + \dot{\theta}\cos\psi\sin\psi] + [\dot{\phi}\sin\theta\cos^2\psi - \dot{\theta}\sin\psi\cos\psi] = \dot{\phi}\sin\theta$$

Thus we have

$$\dot{\phi}\sin\theta = A\sin(\Omega t + \delta)\sin\psi + A\cos(\Omega t + \delta)\cos\psi$$

$$\dot{\phi}\sin\theta = A\cos(\Omega t + \delta + \psi)$$

Plugging this result into (3)

$$\omega_3 = A\frac{\cos\theta}{\sin\theta}\cos(\Omega t + \psi + \delta) + \dot{\psi}$$

Using $\dot{\psi} = -\Omega$ and $\Omega t + \delta + \psi = 0$,

$$\omega_3 = A\frac{\cos\theta}{\sin\theta}\cos(0) - \Omega$$

$$A = (\omega_3 + \Omega)\tan\theta$$

and since $\Omega = \frac{I_3 - I_1}{I_1}\omega_3$

$$A = (\omega_3 + \frac{I_3 - I_1}{I_1}\omega_3)\tan\theta = \frac{I_3}{I_1}\omega_3\tan\theta$$

With this we can solve for the last Euler angle, ϕ,

$$\dot{\phi} = A\frac{\cos(\Omega t + \psi + \delta)}{\sin\theta} = \frac{I_3}{I_1}\omega_3\tan\theta\frac{\cos(0)}{\sin\theta}$$

$$\dot{\phi} = \frac{I_3\omega_3}{I_1 \cos\theta}$$

$$\phi = \frac{I_3\omega_3}{I_1 \cos\theta} t + \phi_0$$

So all together

$$\theta = \theta_0$$

$$\psi(t) = -\Omega t + \psi_0$$

$$\phi(t) = \frac{I_3\omega_3}{I_1 \cos\theta} t + \phi_0$$

Homework 1: # 1, 2, 6, 8, 14, 20

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1. Show that for a single particle with constant mass the equation of motion implies the follwing differential equation for the kinetic energy:

$$\frac{dT}{dt} = \mathbf{F} \cdot \mathbf{v}$$

while if the mass varies with time the corresponding equation is

$$\frac{d(mT)}{dt} = \mathbf{F} \cdot \mathbf{p}.$$

Answer:

$$\frac{dT}{dt} = \frac{d(\frac{1}{2}mv^2)}{dt} = m\mathbf{v} \cdot \dot{\mathbf{v}} = m\mathbf{a} \cdot \mathbf{v} = \mathbf{F} \cdot \mathbf{v}$$

with time variable mass,

$$\frac{d(mT)}{dt} = \frac{d}{dt}(\frac{p^2}{2}) = \mathbf{p} \cdot \dot{\mathbf{p}} = \mathbf{F} \cdot \mathbf{p}.$$

2. Prove that the magnitude R of the position vector for the center of mass from an arbitrary origin is given by the equation:

$$M^2R^2 = M\sum_i m_i r_i^2 - \frac{1}{2}\sum_{i,j} m_i m_j r_{ij}^2.$$

Answer:

$$M\mathbf{R} = \sum m_i \mathbf{r}_i$$

$$M^2\mathbf{R}^2 = \sum_{i,j} m_i m_j \mathbf{r}_i \cdot \mathbf{r}_j$$

Solving for $\mathbf{r}_i \cdot \mathbf{r}_j$ realize that $\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j$. Square $\mathbf{r}_i - \mathbf{r}_j$ and you get

$$r_{ij}^2 = r_i^2 - 2\mathbf{r}_i \cdot \mathbf{r}_j + r_j^2$$

Plug in for $\mathbf{r}_i \cdot \mathbf{r}_j$

$$\mathbf{r}_i \cdot \mathbf{r}_j = \frac{1}{2}(r_i^2 + r_j^2 - r_{ij}^2)$$

$$M^2R^2 = \frac{1}{2}\sum_{i,j} m_i m_j r_i^2 + \frac{1}{2}\sum_{i,j} m_i m_j r_j^2 - \frac{1}{2}\sum_{i,j} m_i m_j r_{ij}^2$$

$$M^2R^2 = \frac{1}{2}M\sum_{i} m_i r_i^2 + \frac{1}{2}M\sum_{j} m_j r_j^2 - \frac{1}{2}\sum_{i,j} m_i m_j r_{ij}^2$$

$$M^2R^2 = M\sum_{i} m_i r_i^2 - \frac{1}{2}\sum_{i,j} m_i m_j r_{ij}^2$$

6. A particle moves in the xy plane under the constraint that its velocity vector is always directed toward a point on the x axis whose abscissa is some given function of time $f(t)$. Show that for $f(t)$ differentiable, but otherwise arbitrary, the constraint is nonholonomic.

Answer:

The abscissa is the x-axis distance from the origin to the point on the x-axis that the velocity vector is aimed at. It has the distance $f(t)$.

I claim that the ratio of the velocity vector components must be equal to the ratio of the vector components of the vector that connects the particle to the point on the x-axis. The directions are the same. The velocity vector components are:

$$v_y = \frac{dy}{dt}$$

$$v_x = \frac{dx}{dt}$$

The vector components of the vector that connects the particle to the point on the x-axis are:

$$V_y = y(t)$$

$$V_x = x(t) - f(t)$$

For these to be the same, then

$$\frac{v_y}{v_x} = \frac{V_y}{V_x}$$

$$\frac{dy}{dx} = \frac{y(t)}{x(t) - f(t)}$$

$$\frac{dy}{y(t)} = \frac{dx}{x(t) - f(t)}$$

This cannot be integrated with $f(t)$ being arbitrary. Thus the constraint is nonholonomic. If the constraint was holonomic then

$$F(x, y, t) = 0$$

would be true. If an arbitrary, but small change of dx, dy, dt was made subject to the constraint then the equation

$$\frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial t}dt = 0$$

would hold. From this it can be seen our constraint equation is actually

$$ydx + (f(t) - x)dy + (0)dt = 0$$

Thus

$$\frac{\partial F}{dt} = 0 \qquad \frac{\partial F}{\partial x} = yI \qquad \frac{\partial F}{\partial y} = (f(t) - x)I$$

where I is our integrating factor, $I(x, y, t)$. The first equation shows $F = F(x, y)$ and the second equation that $I = I(x, y)$. The third equation shows us that all of this is impossible because

$$f(t) = \frac{\partial F}{\partial y}\frac{1}{I(x, y)} + x$$

where $f(t)$ is only dependent on time, but the right side depends only on x and y. There can be no integrating factor for the constraint equation and thus it means this constraint is nonholonomic.

8. If L is a Lagrangian for a system of n degrees of freedom satisfying Lagrange's equations, show by direct substitution that

$$L' = L + \frac{dF(q_1, ..., q_n, t)}{dt}$$

also satisfies Lagrange's equations where F is any arbitrary, but differentiable, function of its arguments.

Answer:

Let's directly substitute L' into Lagrange's equations.

$$\frac{d}{dt}\frac{\partial L'}{\partial \dot{q}} - \frac{\partial L'}{\partial q} = 0$$

$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}}(L + \frac{dF}{dt}) - \frac{\partial}{\partial q}(L + \frac{dF}{dt}) = 0$$

$$\frac{d}{dt}[\frac{\partial L}{\partial \dot{q}} + \frac{\partial}{\partial \dot{q}}\frac{dF}{dt}] - \frac{\partial L}{\partial q} - \frac{\partial}{\partial q}\frac{dF}{dt} = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} + \frac{d}{dt}\frac{\partial}{\partial \dot{q}}\frac{dF}{dt} - \frac{\partial}{\partial q}\frac{dF}{dt} = 0$$

On the left we recognized Lagrange's equations, which we know equal zero. Now to show the terms with F vanish.

$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}}\frac{dF}{dt} - \frac{\partial}{\partial q}\frac{dF}{dt} = 0$$

$$\frac{d}{dt}\frac{\partial \dot{F}}{\partial \dot{q}} = \frac{\partial \dot{F}}{\partial q}$$

This is shown to be true because

$$\frac{\partial \dot{F}}{\partial \dot{q}} = \frac{\partial F}{\partial q}$$

and

$$\frac{d}{dt}\frac{\partial \dot{F}}{\partial \dot{q}} = \frac{d}{dt}\frac{\partial F}{\partial q}$$

$$= \frac{\partial}{\partial t}\frac{\partial F}{\partial q} + \frac{\partial}{\partial q}\frac{\partial F}{\partial q}\dot{q}$$

$$= \frac{\partial}{\partial q}[\frac{\partial F}{\partial t} + \frac{\partial F}{\partial q}\dot{q}] = \frac{\partial \dot{F}}{\partial q}$$

Thus as Goldstein reminded us, $L = T - V$ is a suitable Lagrangian, but it is not the only Lagrangian for a given system.

14. Two points of mass m are joined by a rigid weightless rod of length l, the center of which is constrained to move on a circle of radius a. Express the kinetic energy in generalized coordinates.

Answer:

$$T = T_1 + T_2$$

Where T_1 equals the kinetic energy of the center of mass, and T_2 is the kinetic energy about the center of mass. I will keep these two parts separate.

Solve for T_1 first, its the easiest:

$$T_1 = \frac{1}{2}Mv_{cm}^2 = \frac{1}{2}(2m)(a\dot{\psi})^2 = ma^2\dot{\psi}^2$$

Solve for T_2, realizing that the rigid rod is probably not restricted to just the X-Y plane. The Z-axis adds more complexity to the problem.

$$T_2 = \frac{1}{2}Mv^2 = mv^2$$

Solve for v^2 about the center of mass. The angle ϕ will be the angle in the x-y plane, while the angle θ will be the angle from the z-axis.

If $\theta = 90^o$ and $\phi = 0^o$ then $x = l/2$ so:

$$x = \frac{l}{2}\sin\theta\cos\phi$$

If $\theta = 90^o$ and $\phi = 90^o$ then $y = l/2$ so:

$$y = \frac{l}{2}\sin\theta\sin\phi$$

If $\theta = 0^o$, then $z = l/2$ so:

$$z = \frac{l}{2}\cos\theta$$

Find v^2:

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = v^2$$

$$\dot{x} = \frac{l}{2}(\cos\phi\cos\theta\dot{\theta} - \sin\theta\sin\phi\dot{\phi})$$

$$\dot{y} = \frac{1}{2}(\sin\phi\cos\theta\dot{\theta} + \sin\theta\cos\phi\dot{\phi})$$

$$\dot{z} = -\frac{l}{2}\sin\theta\dot{\theta}$$

Carefully square each:

$$\dot{x}^2 = \frac{l^2}{4}\cos^2\phi\cos^2\theta\dot{\theta}^2 - 2\frac{l}{2}\sin\theta\sin\phi\dot{\phi}\frac{l}{2}\cos\phi\cos\theta\dot{\theta} + \frac{l^2}{4}\sin^2\theta\sin^2\phi\dot{\phi}^2$$

$$\dot{y}^2 = \frac{l^2}{4}\sin^2\phi\cos^2\theta\dot{\theta}^2 + 2\frac{l}{2}\sin\theta\cos\phi\dot{\phi}\frac{l}{2}\sin\phi\cos\theta\dot{\theta} + \frac{l^2}{4}\sin^2\theta\cos^2\phi\dot{\phi}^2$$

$$\dot{z}^2 = \frac{l^2}{4}\sin^2\theta\dot{\theta}^2$$

Now add, striking out the middle terms:

$$\dot{x}^2+\dot{y}^2+\dot{z}^2 = \frac{l^2}{4}[\cos^2\phi\cos^2\theta\dot{\theta}^2+\sin^2\theta\sin^2\phi\dot{\phi}^2+\sin^2\phi\cos^2\theta\dot{\theta}^2+\sin^2\theta\cos^2\phi\dot{\phi}^2+\sin^2\theta\dot{\theta}^2]$$

Pull the first and third terms inside the brackets together, and pull the second and fourth terms together as well:

$$v^2 = \frac{l^2}{4}[\cos^2\theta\dot{\theta}^2(\cos^2\phi+\sin^2\phi)+\sin^2\theta\dot{\phi}^2(\sin^2\phi+\cos^2\phi)+\sin^2\theta\dot{\theta}^2]$$

$$v^2 = \frac{l^2}{4}(\cos^2\theta\dot{\theta}^2+\sin^2\theta\dot{\theta}^2+\sin^2\theta\dot{\phi}^2)$$

$$v^2 = \frac{l^2}{4}(\dot{\theta}^2+\sin^2\theta\dot{\phi}^2)$$

Now that we finally have v^2 we can plug this into T_2

$$T = T_1 + T_2 = ma^2\dot{\psi}^2 + m\frac{l^2}{4}(\dot{\theta}^2+\sin^2\theta\dot{\phi}^2)$$

I want to emphasize again that T_1 is the kinetic energy of the total mass around the center of the circle while T_2 is the kinetic energy of the masses about the center of mass.

20. A particle of mass m moves in one dimension such that it has the Lagrangian

$$L = \frac{m^2\dot{x}^4}{12} + m\dot{x}^2V(x) - V_2(x)$$

where V is some differentiable function of x. Find the equation of motion for $x(t)$ and describe the physical nature of the system on the basis of this system.

Answer:

Correcting for error,

$$L = \frac{m^2\dot{x}^4}{12} + m\dot{x}^2V(x) - V^2(x)$$

Finding the equations of motion from Euler-Lagrange formulation:

$$\frac{\partial L}{\partial x} = +m\dot{x}^2V'(x) - 2V(x)V'(x)$$

$$\frac{\partial L}{\partial \dot{x}} = +\frac{m^2\dot{x}^3}{3} + 2m\dot{x}V(x)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = m^2\dot{x}^2\ddot{x} + 2mV(x)\ddot{x} + 2m\dot{x}V'(x)\dot{x}$$

Thus

$$-m\dot{x}^2V' + 2VV' + m^2\dot{x}^2\ddot{x} + 2mV\ddot{x} + 2m\dot{x}V'(x)\dot{x} = 0$$

$$m\dot{x}^2V' + 2VV' + m^2\dot{x}^2\ddot{x} + 2mV\ddot{x} = 0$$

is our equation of motion. But we want to interpret it. So lets make it look like it has useful terms in it, like kinetic energy and force. This can be done by dividing by 2 and separating out $\frac{1}{2}mv^2$ and ma's.

$$\frac{m\dot{x}^2}{2}V' + VV' + \frac{m\dot{x}^2}{2}m\ddot{x} + m\ddot{x}V = 0$$

Pull V' terms together and $m\ddot{x}$ terms together:

$$(\frac{m\dot{x}^2}{2} + V)V' + m\ddot{x}(\frac{m\dot{x}^2}{2} + V) = 0$$

Therefore:

$$(\frac{m\dot{x}^2}{2} + V)(m\ddot{x} + V') = 0$$

Now this looks like $E \cdot E' = 0$ because $E = \frac{m\dot{x}^2}{2} + V(x)$. That would mean

$$\frac{d}{dt}E^2 = 2EE' = 0$$

Which reveals that E^2 is a constant. If we look at $t = 0$ and the starting energy of the particle, then we will notice that if $E = 0$ at $t = 0$ then $E = 0$ for all other times. If $E \neq 0$ at $t = 0$ then $E \neq 0$ all other times while $m\ddot{x} + V' = 0$.

Homework 3: # 2.13, 2.14

Michael Good

Sept 10, 2004

2.13 A heavy particle is placed at the top of a vertical hoop. Calculate the reaction of the hoop on the particle by means of the Lagrange's undetermined multipliers and Lagrange's equations. Find the height at which the particle falls off.

Answer:

The Lagrangian is

$$L = T - V \qquad \Rightarrow \qquad L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - mgr\cos\theta$$

Where r is the distance the particle is away from the center of the hoop. The particle will eventually fall off but while its on the hoop, r will equal the radius of the hoop, a. This will be the constraint on the particle. Here when $\theta = 0$, (at the top of the hoop) potential energy is mgr, and when $\theta = 90^o$ (at half of the hoop) potential energy is zero. Using Lagrange's equations with undetermined multipliers,

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} + \sum_k \lambda \frac{\partial f_k}{\partial q_j} = 0$$

with our equation of constraint, $f = r = a$ as long as the particle is touching the hoop. Solving for the motion:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = m\ddot{r}$$

$$\frac{\partial L}{\partial r} = mr\dot{\theta}^2 - mg\cos\theta$$

$$\lambda\frac{\partial f_r}{\partial r} = \lambda * 1$$

thus

$$-m\ddot{r} + mr\dot{\theta}^2 - mg\cos\theta + \lambda = 0$$

solving for the other equation of motion,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta}$$
$$\frac{\partial L}{\partial \theta} = mgr\sin\theta$$
$$\lambda\frac{\partial f_\theta}{\partial \theta} = \lambda * 0$$

thus

$$-mr^2\ddot{\theta} - 2mr\dot{r}\dot{\theta} + mgr\sin\theta = 0$$

The equations of motion together are:

$$-m\ddot{r} + mr\dot{\theta}^2 - mg\cos\theta + \lambda = 0$$

$$-mr^2\ddot{\theta} - 2mr\dot{r}\dot{\theta} + mgr\sin\theta = 0$$

To find the height at which the particle drops off, λ can be found in terms of θ. The force of constraint is λ and $\lambda = 0$ when the particle is no longer under the influence of the force of the hoop. So finding λ in terms of θ and setting λ to zero will give us the magic angle that the particle falls off. With the angle we can find the height above the ground or above the center of the hoop that the particle stops maintaining contact with the hoop.

With the constraint, the equations of motion become,

$$ma\dot{\theta}^2 - mg\cos\theta + \lambda = 0$$
$$-ma^2\ddot{\theta} + mga\sin\theta = 0$$

Solving for θ, the m's cancel and 1 a cancels, we are left with

$$\frac{g}{a}\sin\theta = \ddot{\theta}$$

solving this and noting that

$$\dot{\theta}d\dot{\theta} = \ddot{\theta}d\theta$$

by the 'conservation of dots' law Engel has mentioned :), or by

$$\ddot{\theta} = \frac{d}{dt}\frac{d\theta}{dt} = \frac{d\dot{\theta}}{dt} = \frac{d\dot{\theta}}{d\theta}\frac{d\theta}{dt} = \dot{\theta}\frac{d\dot{\theta}}{d\theta}$$
$$\int \frac{g}{a}\sin\theta d\theta = \int \dot{\theta}d\dot{\theta}$$

$$-\frac{g}{a}\cos\theta = \frac{\dot{\theta}^2}{2} + constant$$

The constant is easily found because at the top of the hoop, $\theta = 0$ and $\dot{\theta} = 0$ at $t = 0$ so,

$$-\frac{2g}{a}\cos\theta + \frac{2g}{a} = \dot{\theta}^2$$

Plug this into our first equation of motion to get an equation dependent only on θ and λ

$$ma[-\frac{2g}{a}\cos\theta + \frac{2g}{a}] - mg\cos\theta = -\lambda$$

$$-3mg\cos\theta + 2mg = -\lambda$$

Setting $\lambda = 0$ because this is at the point where the particle feels no force from the hoop, and θ_0 equals

$$\theta_0 = \cos^{-1}(\frac{2}{3}) = 48.2^o$$

And if our origin is at the center of the hope, then the height that it stops touching the hoop is just $R\cos\theta_0$ or

$$h = R\cos(\cos^{-1}\frac{2}{3}) = \frac{2}{3}R$$

.

If we say the hoop is a fully circular and somehow fixed with the origin at the bottom of the hoop, then we have just moved down by R and the new height is

$$H = R + \frac{2}{3}R = \frac{5}{3}R$$

2.14 A uniform hoop of mass m and radius r rolls without slipping on a fixed cylinder of radius R as shown in figure. The only external force is that of gravity. If the smaller cylinder starts rolling from rest on top of the bigger cylinder, use the method of Lagrange multipliers to find the point at which the hoop falls off the cylinder.

Answer:

Two equations of constraint:

$$\rho = r + R \qquad r(\phi - \theta) = R\theta$$

My generalized coordinates are ρ, θ, and ϕ. The first equation comes from the fact that as long as the hoop is touching the cylinder the center of mass of the hoop is exactly $r + R$ away from the center of the cylinder. I'm calling it f_1. The second one comes from no slipping:

$$r\phi = s \quad \rightarrow \quad s = (R + r)\theta$$

$$r\phi - r\theta = R\theta$$

$$r(\phi - \theta) = R\theta$$

Where θ is the angle ρ makes with the vertical and ϕ is the angle r makes with the vertical. I'm calling this equation f_2.

$$f_1 = \rho - r - R = 0 \qquad f_2 = R\theta - r\phi + r\theta = 0$$

The Lagrangian is $T - V$ where T is the kinetic energy of the hoop about the cylinder and the kinetic energy of the hoop about its center of mass. The potential energy is the height above the center of the cylinder. Therefore

$$L = \frac{m}{2}(\dot{\rho}^2 + \rho^2\dot{\theta}^2 + r^2\dot{\phi}^2) - mg\rho\cos\theta$$

Solving for the equations of motion:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\rho}} - \frac{\partial L}{\partial \rho} = \sum_k \lambda_k \frac{\partial f_k}{\partial \rho}$$

$$m\ddot{\rho} - m\rho\dot{\theta}^2 + mg\cos\theta = \lambda_1\frac{\partial f_1}{\partial \rho} + \lambda_2\frac{\partial f_2}{\partial \rho}$$

$$m\ddot{\rho} - m\rho\dot{\theta}^2 + mg\cos\theta = \lambda_1 \tag{1}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = \sum_k \lambda_k \frac{\partial f_k}{\partial \theta}$$

$$\frac{d}{dt}(m\rho^2\dot{\theta}) - (-mg\rho\sin\theta) = \lambda_1\frac{\partial f_1}{\partial \theta} + \lambda_2\frac{\partial f_2}{\partial \theta}$$

$$m\rho^2\ddot{\theta} + \dot{\theta}2m\rho\dot{\rho} + mg\rho\sin\theta = \lambda_1(0) + \lambda_2(R + r)$$

$$m\rho^2\ddot{\theta} + 2m\rho\dot{\rho}\dot{\theta} + mg\rho\sin\theta = \lambda_2(R + r) \tag{2}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} - \frac{\partial L}{\partial \phi} = \sum_k \lambda_k \frac{\partial f_k}{\partial \phi}$$

$$\frac{d}{dt}(mr^2\dot{\phi}) - 0 = \lambda_1\frac{\partial f_1}{\partial \phi} + \lambda_2\frac{\partial f_2}{\partial \phi}$$

$$mr^2\ddot{\phi} = -\lambda_2 r \tag{3}$$

I want the angle θ. This will tell me the point that the hoop drops off the cylinder. So I'm going to apply the constraints to my equations of motion, attempt to get an equation for θ, and then set λ_1 equal to zero because that will be when the force of the cylinder on the hoop is zero. This will tell me the value of θ. Looking for an equation in terms of only θ and λ_1 will put me in the right position.
The constraints tell me:

$$\rho = r + R \quad \rightarrow \quad \dot{\rho} = \ddot{\rho} = 0$$

$$\phi = \frac{R+r}{r}\theta \quad \rightarrow \quad \dot{\phi} = \frac{R+r}{r}\dot{\theta} \quad \rightarrow \quad \ddot{\phi} = \frac{R+r}{r}\ddot{\theta}$$

Solving (3) using the constraints,

$$mr^2\ddot{\phi} = -\lambda_2 r$$

$$\ddot{\theta} = -\frac{\lambda_2}{m(R+r)} \tag{4}$$

Solving (2) using the constraints,

$$m(R+r)\ddot{\theta} + mg\sin\theta = \lambda_2$$

$$\ddot{\theta} = \frac{\lambda_2 - mg\sin\theta}{m(R+r)} \tag{5}$$

Setting (4) = (5)

$$-\lambda_2 = \lambda_2 - mg\sin\theta$$

$$\lambda_2 = \frac{mg}{2}\sin\theta \tag{6}$$

Plugging (6) into (4) yields a differential equation for θ

$$\ddot{\theta} = \frac{-g}{2(R+r)}\sin\theta$$

If I solve this for $\dot{\theta}^2$ I can place it in equation of motion (1) and have an expression in terms of θ and λ_1. This differential equation can be solved by trying this:

$$\dot{\theta}^2 = A + B\cos\theta$$

Taking the derivative,

$$2\dot{\theta}\ddot{\theta} = -B\sin\theta\dot{\theta}$$

$$\ddot{\theta} = -\frac{B}{2}\sin\theta$$

Thus

$$B = -\frac{q}{R+r}$$

From initial conditions, $\theta = 0$, $\dot{\theta} = 0$ at $t = 0$ we have A:

$$A = -B \quad \rightarrow \quad A = \frac{q}{R+r}$$

Therefore

$$\dot{\theta}^2 = \frac{q}{R+r} - \frac{q}{R+r}\cos\theta$$

Now we are in a position to plug this into equation of motion (1) and have the equation in terms of θ and λ_1

$$-m(R+r)(\frac{q}{R+r} - \frac{q}{R+r}\cos\theta) + mg\cos\theta = \lambda_1$$

$$-mg + 2mg\cos\theta = \lambda_1$$

$$mg(2\cos\theta - 1) = \lambda_1$$

Setting the force of constraint equal to zero will give us the angle that the hoop no longer feels a force from the cylinder:

$$2\cos\theta_0 - 1 = 0$$

$$\cos\theta_0 = \frac{1}{2} \quad \rightarrow \quad \theta_0 = 60^o$$

With our origin at the center of the cylinder, the height that the center of mass of the hoop falls off is

$$h_{cm} = \rho\cos(60^o) = \frac{1}{2}\rho$$

Or if you prefer the height that the hoop's surface stops contact with cylinder:

$$h = \frac{1}{2}R$$

.

Homework 4: # 2.18, 2.21, 3.13, 3.14, 3.20

Michael Good

Sept 20, 2004

2.18 A point mass is constrained to move on a massless hoop of radius a fixed in a vertical plane that rotates about its vertical symmetry axis with constant angular speed ω. Obtain the Lagrange equations of motion assuming the only external forces arise from gravity. What are the constants of motion? Show that if ω is greater than a critical value ω_0, there can be a solution in which the particle remains stationary on the hoop at a point other than the bottom, but if $\omega < \omega_0$, the only stationary point for the particle is at the bottom of the hoop. What is the value of ω_0?

Answer:

To obtain the equations of motion, we need to find the Lagrangian. We only need one generalized coordinate, because the radius of the hoop is constant, and the point mass is constrained to this radius, while the angular velocity, w is constant as well.

$$L = \frac{1}{2}ma^2(\dot{\theta}^2 + \omega^2 \sin^2\theta) - mga\cos\theta$$

Where the kinetic energy is found by spherical symmetry, and the potential energy is considered negative at the bottom of the hoop, and zero where the vertical is at the center of the hoop. My θ is the angle from the z-axis, and a is the radius.

The equations of motion are then:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$$

$$ma^2\ddot{\theta} = ma^2\omega^2 \sin\theta\cos\theta + mga\sin\theta$$

We see that the Lagrangian does not explicitly depend on time therefore the energy function, h, is conserved (Goldstein page 61).

$$h = \dot{\theta}\frac{\partial L}{\partial \dot{\theta}} - L$$

$$h = \dot{\theta}ma^2\dot{\theta} - \frac{1}{2}ma^2(\dot{\theta}^2 + \omega^2\sin^2\theta) - mga\cos\theta$$

This simplifies to:

$$h = \frac{1}{2}ma^2\dot{\theta} - (\frac{1}{2}ma^2\omega^2 \sin^2\theta - mga\cos\theta)$$

Because the 'energy function' has an identical value to the Hamiltonian, the effective potential is the second term,

$$V_{eff} = mga\cos\theta - \frac{1}{2}ma^2\omega^2\sin^2\theta$$

The partial of V_{eff} with respect to θ set equal to zero should give us a stationary point.

$$\frac{\partial V_{eff}}{\partial\theta} = mga\sin\theta + ma^2\omega^2\sin\theta\cos\theta = 0$$

$$ma\sin\theta(g + a\omega^2\cos\theta) = 0$$

This yields three values for θ to obtain a stationary point,

$$\theta = 0 \quad \theta = \pi \quad \theta = \arccos(-\frac{g}{a\omega^2})$$

At the top, the bottom, and some angle that suggests a critical value of ω.

$$\omega_0 = \sqrt{\frac{g}{a}}$$

The top of the hoop is unstable, but at the bottom we have a different story. If I set $\omega = \omega_0$ and graph the potential, the only stable minimum is at $\theta = \pi$, the bottom. Therefore anything $\omega < \omega_0$, $\theta = \pi$ is stable, and is the only stationary point for the particle.

If we speed up this hoop, $\omega > \omega_0$, our angle

$$\theta = \arccos(-\frac{\omega_0^2}{\omega^2})$$

is stable and $\theta = \pi$ becomes unstable. So the point mass moves up the hoop, to a nice place where it is swung around and maintains a stationary orbit.

2.21 A carriage runs along rails on a rigid beam, as shown in the figure below. The carriage is attached to one end of a spring of equilibrium length r_0 and force constant k, whose other end is fixed on the beam. On the carriage, another set of rails is perpendicular to the first along which a particle of mass m moves, held by a spring fixed on the beam, of force constant k and zero equilibrium length. Beam, rails, springs, and carriage are assumed to have zero mass. The whole system is forced to move in a plane about the point of attachment of the first spring, with a constant angular speed ω. The length of the second spring is at all times considered small compared to r_0.

- What is the energy of the system? Is it conserved?
- Using generalized coordinates in the laboratory system, what is the Jacobi integral for the system? Is it conserved?
- In terms of the generalized coordinates relative to a system rotating with the angular speed ω, what is the Lagrangian? What is the Jacobi integral? Is it conserved? Discuss the relationship between the two Jacobi integrals.

Answer:

Energy of the system is found by the addition of kinetic and potential parts. The kinetic, in the lab frame, (x, y), using Cartesian coordinates is

$$T(x,y) = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

Potential energy is harder to write in lab frame. In the rotating frame, the system looks stationary, and its potential energy is easy to write down. I'll use (r, l) to denote the rotating frame coordinates. The potential, in the rotating frame is

$$V(r,l) = \frac{1}{2}k(r^2 + l^2)$$

Where r is simply the distance stretched from equilibrium for the large spring. Since the small spring has zero equilbrium length, then the potential energy for it is just $\frac{1}{2}kl^2$.

The energy needs to be written down fully in one frame or the other, so I'll need a pair of transformation equations relating the two frames. That is, relating (x, y) to (r, l). Solving for them, by drawing a diagram, yields

$$x = (r_0 + r)\cos\omega t - l\sin\omega t$$
$$y = (r_0 + r)\sin\omega t + l\cos\omega t$$

Manipulating these so I may find $r(x, y)$ and $l(x, y)$ so as to write the stubborn potential energy in terms of the lab frame is done with some algebra.

Multiplying x by $\cos\omega t$ and y by $\sin\omega t$, adding the two equations and solving for r yields

$$r = x\cos\omega t + y\sin\omega t - r_0$$

Multiplying x by sin and y by cos, adding and solving for l yields

$$l = -x\sin\omega t + y\cos\omega t$$

Plugging these values into the potential energy to express it in terms of the lab frame leaves us with

$$E(x,y) = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}k((x\cos\omega t + y\sin\omega t - r_0)^2 + (-x\sin\omega t + y\cos\omega t)^2)$$

This energy is explicitly dependent on time. Thus it is NOT conserved in the lab frame. $E(x,y)$ is not conserved.

In the rotating frame this may be a different story. To find $E(r,l)$ we are lucky to have an easy potential energy term, but now our kinetic energy is giving us problems. We need

$$E(r,l) = T(r,l) + \frac{1}{2}k(l^2 + r^2)$$

Where in the laboratory frame, $T(x,y) = \frac{1}{2}(\dot{x}^2 + \dot{y}^2)$. Taking derivatives of x and y yield

$$\dot{x} = -\omega(r_0 + r)\sin\omega t + \dot{r}\cos\omega t - l\omega\cos\omega t - \dot{l}\sin\omega t$$
$$\dot{y} = \omega(r_0 + r)\cos\omega t + \dot{r}\sin\omega t - l\omega\sin\omega t - \dot{l}\cos\omega t$$

Squaring both and adding them yields

$$\dot{x}^2 + \dot{y}^2 = \omega^2(r_0 + r)^2\dot{r}^2 + l^2\omega^2 + \dot{l}^2 + C.T.$$

Where cross terms, C.T. are

$$C.T. = 2\omega(r_0 + r)\dot{l} - 2\dot{r}l\omega$$

For kinetic energy we know have

$$T(r,l) = \frac{1}{2}m(\omega^2(r_0 + r)^2\dot{r}^2 + l^2\omega^2 + \dot{l}^2 + 2\omega(r_0 + r)\dot{l} - 2\dot{r}l\omega)$$

Collecting terms

$$T(r,l) = \frac{1}{2}m(\omega^2(r_0 + r + \frac{\dot{l}}{\omega})^2 + (\dot{r} - l\omega)^2)$$

Thus

$$E(r,l) = \frac{1}{2}m(\omega^2(r_0 + r + \frac{\dot{l}}{\omega})^2 + (\dot{r} - l\omega)^2) + \frac{1}{2}k(l^2 + r^2)$$

This has no explicit time dependence, therefore energy in the rotating frame is conserved. $E(r,l)$ is conserved.

In the laboratory frame, the Lagrangian is just $T(x,y) - V(x,y)$.

$$L(x,y) = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - V(x,y)$$

Where

$$V(x,y) = \frac{1}{2}k((x\cos\omega t + y\sin\omega t - r_0)^2 + (-x\sin\omega t + y\cos\omega t)^2)$$

The Jacobi integral, or energy function is

$$h = \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L$$

We have

$$h = \dot{x}\frac{\partial L}{\partial \dot{x}} + \dot{y}\frac{\partial L}{\partial \dot{y}} - L(x,y)$$

$$h = \dot{x}m\dot{x} + \dot{y}m\dot{y} - \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + V(x,y)$$

Notice that $V(x,y)$ does not have any dependence on $\dot{x}$ or $\dot{y}$. Bringing it together

$$h = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}k((x\cos\omega t + y\sin\omega t - r_0)^2 + (-x\sin\omega t + y\cos\omega t)^2)$$

This is equal to the energy.

$$h(x,y) = E(x,y)$$

Because it is dependent on time,

$$\frac{d}{dt}h = -\frac{\partial L}{\partial t} \neq 0$$

we know $h(x,y)$ is not conserved in the lab frame.

For the rotating frame, the Lagrangian is

$$L(r,l) = T(r,l) - \frac{1}{2}k(r^2 + l^2)$$

Where

$$T(r,l) = \frac{1}{2}m(\omega^2(r_0 + r + \frac{\dot{l}}{\omega})^2 + (\dot{r} - l\omega)^2)$$

The energy function, or Jacobi integral is

$$h(r,l) = \dot{r}\frac{\partial L}{\partial \dot{r}} + \dot{l}\frac{\partial L}{\partial \dot{l}} - L(r,l)$$

$$h(r,l) = \dot{r}m(\dot{r} - l\omega) + \dot{l}m\omega(r_0 + r + \frac{\dot{l}}{\omega}) - L(r,l)$$

Collecting terms, with some heavy algebra

$$h = (\dot{r}-l\omega)(m\dot{r}-\frac{1}{2}m(\dot{r}-l\omega))+(r_0+r+\frac{\dot{l}}{\omega})(m\omega\dot{l}-\frac{1}{2}m\omega^2(r_0+r+\frac{\dot{l}}{\omega}))+\frac{1}{2}k(l^2+r^2)$$

$$h = (\dot{r} - l\omega)(\frac{m\dot{r}}{2} + \frac{1}{2}ml\omega) + (r_0 + r + \frac{\dot{l}}{\omega})(\frac{1}{2}m\omega\dot{l} - \frac{1}{2}m\omega^2(r_0 + r)) + \frac{1}{2}k(l^2 + r^2)$$

More algebraic manipulation in order to get terms that look like kinetic energy,

$$h = \frac{1}{2}m(\dot{r}^2+\dot{l}^2)+\frac{1}{2}k(l^2+r^2)+\frac{1}{2}[\dot{r}ml\omega-l\omega m\dot{r}-ml^2\omega^2+(r_0+r)m\omega\dot{l}-m\omega^2(r_0+r)^2-m\omega\dot{l}(r_0+r)]$$

Yields

$$h(r,l) = \frac{1}{2}m(\dot{r}^2 + \dot{l}^2) + \frac{1}{2}k(l^2 + r^2) - \frac{1}{2}m\omega^2(l^2 + (r_0 + r)^2)$$

This has no time dependence, and this nice way of writing it reveals an energy term of rotation in the lab frame that can't be seen in the rotating frame. It is of the from $E = -\frac{1}{2}I\omega^2$.

$$\frac{d}{dt}h = -\frac{\partial L}{\partial t} = 0$$

We have $h(r,l)$ conserved in the rotating frame.

3.13

- Show that if a particle describes a circular orbit under the influence of an attractive central force directed toward a point on the circle, then the force varies as the inverse-fifth power of the distance.
- Show that for orbit described the total energy of the particle is zero.
- Find the period of the motion.
- Find $\dot{x}$, $\dot{y}$ and v as a function of angle around the circle and show that all three quantities are infinite as the particle goes through the center of force.

Answer:

Using the differential equation of the orbit, equation (3.34) in Goldstein,

$$\frac{d^2}{d\theta^2}u + u = -\frac{m}{l^2}\frac{d}{du}V(\frac{1}{u})$$

Where $r = 1/u$ and with the origin at a point on the circle, a triangle drawn with r being the distance the mass is away from the origin will reveal

$$r = 2R\cos\theta$$

$$u = \frac{1}{2R\cos\theta}$$

Plugging this in and taking the derivative twice,

$$\frac{d}{d\theta}u = \frac{1}{2R}[-\cos^{-2}\theta(-\sin\theta)] = \frac{\sin\theta}{2R\cos^2\theta}$$

The derivative of this is

$$\frac{d}{d\theta}\frac{\sin\theta}{2R\cos^2\theta} = \frac{1}{2R}[\sin\theta(-2\cos^{-3}\theta)(-\sin\theta) + \cos^{-2}\theta\cos\theta]$$

Thus

$$\frac{d^2}{d\theta^2}u = \frac{1}{2R}[\frac{2\sin^2\theta}{\cos^3\theta} + \frac{\cos^2\theta}{\cos^3\theta}] = \frac{1+\sin^2\theta}{2R\cos^3\theta}$$

$$\frac{d^2}{d\theta^2}u + u = \frac{1+\sin^2\theta}{2R\cos^3\theta} + \frac{\cos^2\theta}{2R\cos^3\theta} = \frac{2}{2R\cos^3\theta}$$

That is

$$\frac{8R^2}{8R^3\cos^3\theta} = 8R^2u^3$$

Solving for $V(\frac{1}{u})$ by integrating yields,

$$V(\frac{1}{u}) = -\frac{8R^2l^2}{4m}u^4$$

and we have

$$V(r) = -\frac{2l^2R^2}{mr^4}$$

with force equal to

$$f(r) = -\frac{d}{dr}V(r) = -\frac{8l^2R^2}{mr^5}$$

This force is inversely proportional to r^5.

Is the energy zero? Well, we know $V(r)$, lets find $T(r)$ and hope its the negative of $V(r)$.

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$$

Where

$$r = 2R\cos\theta \qquad \rightarrow \qquad \dot{r} = -2R\sin\theta\dot{\theta}$$

$$r^2 = 4R^2\cos^2\theta \qquad \dot{r}^2 = 4R^2\sin^2\dot{\theta}^2$$

So, plugging these in,

$$T = \frac{1}{2}m(4R^2\sin^2\theta\dot{\theta}^2 + 4R^2\cos^2\theta\dot{\theta}^2)$$

$$T = \frac{m4R\dot{\theta}^2}{2} = 2mR^2\dot{\theta}^2$$

Put this in terms of angular momentum, l,

$$l = mr^2\dot{\theta}$$

$$l^2 = m^2r^4\dot{\theta}^2$$

$$T = 2mR^2\dot{\theta}^2 \quad \rightarrow \quad T = \frac{2R^2l^2}{mr^4}$$

Which shows that

$$E = T + V = \frac{2R^2l^2}{mr^4} - \frac{2R^2l^2}{mr^4} = 0$$

the total energy is zero.

The period of the motion can be thought of in terms of θ as r spans from $\theta = -\frac{\pi}{2}$ to $\theta = \frac{\pi}{2}$.

$$P = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{dt}{d\theta} d\theta$$

This is

$$P = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta}{\dot{\theta}}$$

Because $\dot{\theta} = l/mr^2$ in terms of angular momentum, we have

$$P = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{mr^2}{l} d\theta$$

$$P = \frac{m}{l} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r^2 d\theta$$

From above we have r^2

$$P = \frac{m}{l} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4R^2 \cos^2 \theta d\theta$$

$$P = \frac{4mR^2}{l} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 \theta d\theta = \frac{4mR^2}{l} \left(\frac{\theta}{2} + \frac{1}{4} \sin 2\theta \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \right) = \frac{4mR^2}{l} \left(\frac{\pi}{4} + \frac{\pi}{4} \right)$$

And finally,

$$P = \frac{2m\pi R^2}{l}$$

For $\dot{x}$, $\dot{y}$, and v as a function of angle, it can be shown that all three quantities are infinite as particle goes through the center of force. Remembering that $r = 2R\cos\theta$,

$$x = r\cos\theta = 2R\cos^2\theta$$

$$y = r\sin\theta = 2R\cos\theta\sin\theta = R\sin 2\theta$$

Finding their derivatives,

$$\dot{x} = -4R\cos\theta\sin\theta = -2R\dot{\theta}\sin 2\theta$$

$$\dot{y} = 2R\dot{\theta}\cos 2\theta$$

$$v = \sqrt{\dot{x}^2 + \dot{y}^2} = 2R\dot{\theta}$$

What is $\dot{\theta}$? In terms of angular momentum we remember

$$l = mr^2\dot{\theta}$$

Plugging in our r, and solving for $\dot{\theta}$

$$\dot{\theta} = \frac{l}{4mR^2\cos^2\theta}$$

As we got closer to the origin, θ becomes close to $\pm\frac{\pi}{2}$.

$$\theta = \pm(\frac{\pi}{2} - \delta)$$

Note that as

$$\delta \to 0 \qquad \theta \to \pm\frac{\pi}{2} \qquad \dot{\theta} \to \infty$$

All $\dot{x}$, $\dot{y}$ and v are directly proportional to the $\dot{\theta}$ term. The $\dot{x}$ may be questionable at first because it has a $\sin 2\theta$ and when $\sin 2\theta \to 0$ as $\theta \to \pi/2$ we may be left with $\infty * 0$. But looking closely at $\dot{\theta}$ we can tell that

$$\dot{x} = \frac{-4Rl\cos\theta\sin\theta}{4mR^2\cos^2\theta} = -\frac{l}{mR}\tan\theta$$

$$\tan\theta \to \infty \quad as \quad \theta \to \pm\frac{\pi}{2}$$

2.14

- For circular and parabolic orbits in an attractive $1/r$ potential having the same angular momentum, show that perihelion distance of the parabola is one-half the radius of the circle.
- Prove that in the same central force as above, the speed of a particle at any point in a parabolic orbit is $\sqrt{2}$ times the speed in a circular orbit passing through the same point.

Answer:

Using the equation of orbit, Goldstein equation 3.55,

$$\frac{1}{r} = \frac{mk}{l^2}[1 + \epsilon\cos(\theta - \theta')]$$

we have for the circle, $\epsilon = 0$

$$\frac{1}{r_c} = \frac{mk}{l^2} \to r_c = \frac{l^2}{mk}$$

For the parabola, $\epsilon = 1$

$$\frac{1}{r_p} = \frac{mk}{l^2}(1+1) \to r_p = \frac{l^2}{2mk}$$

So

$$r_p = \frac{r_c}{2}$$

The speed of a particle in a circular orbit is

$$v_c^2 = r^2\dot{\theta}^2 \quad \rightarrow \quad v_c^2 = r^2(\frac{l^2}{m^2r^4}) \quad \rightarrow \quad v_c = \frac{l}{mr}$$

In terms of k, this is equal to

$$\frac{l}{mr} = \frac{\sqrt{mrk}}{mr} = \sqrt{\frac{k}{mr}}$$

The speed of a particle in a parabola can be found by

$$v_p^2 = \dot{r}^2 + r^2\dot{\theta}^2$$

$$\dot{r} = \frac{d}{dt}(\frac{l^2}{mk(1+\cos\theta)}) = \frac{l^2\dot{\theta}}{mk(1+\cos\theta)^2}\sin\theta$$

Solving for v_p,

$$v_p^2 = r^2\dot{\theta}^2(\frac{sin^2\theta}{(1+\cos\theta)^2} + 1)$$

$$v_p^2 = r^2\dot{\theta}^2(\frac{2+2\cos\theta}{(1+\cos\theta)^2})$$

$$v_p^2 = \frac{2r^2\dot{\theta}^2}{1+\cos\theta}$$

Using r for a parabola from Goldstein's (3.55), and not forgetting that $k = l^2/mr$,

$$r = \frac{l^2}{mk(1+\cos\theta)} \rightarrow \dot{\theta}^2 = \frac{l^2}{m^2r^4}$$

we have

$$v_p^2 = \frac{2r^2l^2mkr}{m^2r^4l^2} \rightarrow v_p^2 = \frac{2k}{mr}$$

For the speed of the parabola, we then have

$$v_p = \sqrt{2}\sqrt{\frac{k}{mr}}$$

Thus

$$v_p = \sqrt{2}v_c$$

20. A uniform distribution of dust in the solar system adds to the gravitational attraction of the Sun on a planet an additional force

$$F = -mCr$$

where m is the mas of the planet, C is a constant proportional to the gravitational constant and the density of the dust, and r is the radius vector from the Sun to the planet(both considered as points). This additional force is very small compared to the direct Sun-planet gravitational force.

- Calculate the period for a circular orbit of radius r_0 of the planet in this combined field.
- Calculate the period of radial oscillations for slight disturbances from the circular orbit.
- Show that nearly circular orbits can be approximated by a precessing ellipse and find the precession frequency. Is the precession in the same or opposite direction to the orbital angular velocity?

Answer:

The equation for period is

$$T = \frac{2\pi}{\dot{\theta}}$$

For a circular orbit,

$$\dot{\theta} = \frac{l}{mr^2}$$

Thus

$$T = \frac{2\pi mr^2}{l}$$

Goldstein's equation after (3.58):

$$\frac{k}{r_0^2} = \frac{l^2}{mr_0^3}$$

In our case, we have an added force due to the dust,

$$mCr_0 + \frac{k}{r_0^2} = \frac{l^2}{mr_0^3}$$

Solving for l yields

$$l = \sqrt{mr_0k + m^2Cr_0^4}$$

Plugging this in to our period,

$$T = \frac{2\pi m r^2}{\sqrt{m r_0 k + m^2 C r_0^4}} \quad \rightarrow \quad T = \frac{2\pi}{\sqrt{\frac{k}{m r_0^3} + C}}$$

Here the orbital angular velocity is

$$\omega_{orb} = \sqrt{\frac{k}{m r^3} + C}$$

This is nice because if the dust was not there, we would have $C = 0$ and our period would be

$$T_0 = \frac{2\pi}{\sqrt{\frac{k}{m r_0^2}}} \quad \rightarrow \quad \omega_0 = \sqrt{\frac{k}{m r_0^2}}$$

which agrees with $l = m r_0^2 \omega_0$ and $l = \sqrt{mrk}$.

The period of radial oscillations for slight disturbances from the circular orbit can be calculated by finding β. β is the number of cycles of oscillation that the particle goes through in one complete orbit. Dividing our orbital period by β will give us the period of the oscillations.

$$T_{osc} = \frac{T}{\beta}$$

Equation (3.45) in Goldstein page 90, states that for small deviations from circularity conditions,

$$u \equiv \frac{1}{r} = u_0 + a \cos \beta\theta$$

Substitution of this into the force law gives equation (3.46)

$$\beta^2 = 3 + \frac{r}{f}\frac{df}{dr}\bigg|_{r=r_0}$$

Solve this with $f = mCr + k/r^2$

$$\frac{df}{dr} = -\frac{2k}{r^3} + mC$$

$$\beta^2 = 3 + r\frac{-\frac{2k}{r^3} + mC}{\frac{k}{r^2} + mCr}$$

$$\beta^2 = \frac{\frac{k}{r^2} + 4mCr}{\frac{k}{r^2} + mCr} \quad \rightarrow \quad \beta^2 = \frac{\frac{k}{mr^3} + 4C}{\frac{k}{mr^3} + C}$$

Now

$$T_{osc} = \frac{T}{\beta} \qquad\qquad \beta = \frac{\sqrt{\frac{k}{mr^3} + 4C}}{\sqrt{\frac{k}{mr^3} + C}}$$

Therefore, our period of radial oscillations is

$$T_{osc} = \frac{2\pi}{\sqrt{\frac{k}{mr^3} + 4C}}$$

Here

$$\omega_r = \sqrt{\frac{k}{mr^3} + 4C}$$

A nearly circular orbit can be approximated by a precessing ellipse. The equation for an elliptical orbit is

$$r = \frac{a(1-e^2)}{1 + e\cos(\theta - \theta_0}$$

with $e << 1$, for a nearly circular orbit, a precessing ellipse will hug closely to the circle that would be made by $e = 0$.

To find the precession frequency, I'm going to subtract the orbital angular velocity from the radial angular velocity,

$$\omega_{prec} = \omega_r - \omega_{orb}$$

$$\omega_{prec} = \sqrt{\frac{k}{mr^3} + 4C} - \sqrt{\frac{k}{mr^3} + C}$$

Fixing this up so as to use the binomial expansion,

$$\omega_{prec} = \sqrt{\frac{k}{mr^3}}(\sqrt{1 + \frac{4Cmr^3}{k}} - \sqrt{1 + \frac{Cmr^3}{k}})$$

Using the binomial expansion,

$$\omega_{prec} = \sqrt{\frac{k}{mr^3}}[1 + \frac{2Cmr^3}{k} - (1 + \frac{Cmr^3}{2k})] = \sqrt{\frac{k}{mr^3}}[\frac{2Cmr^3}{k} - \frac{Cmr^3}{2k}]$$

$$\omega_{prec} = \sqrt{\frac{k}{mr^3}}\frac{4Cmr^3 - Cmr^3}{2k} = \frac{3Cmr^3}{2k}\sqrt{\frac{k}{mr^3}} = \frac{3C}{2}\sqrt{\frac{mr^3}{k}}$$

Therefore,

$$\omega_{prec} = \frac{3C}{2\omega_0} \quad \rightarrow \quad f_{prec} = \frac{3C}{4\pi\omega_0}$$

Because the radial oscillations take on a higher angular velocity than the orbital angular velocity, the orbit is very nearly circular but the radial extrema comes a tiny bit more than once per period. This means that the orbit precesses opposite the direction of the orbital motion.

Another way to do it, would be to find change in angle for every oscillation,

$$\Delta\theta = 2\pi - \frac{2\pi}{\beta}$$

Using the ratios,

$$T_{prec} = \frac{2\pi}{\Delta\theta} T_{osc}$$

With some mean algebra, the period of precession is

$$T_{prec} = \frac{4\pi}{1 - \frac{\frac{k}{mr^3}+C}{\frac{k}{mr^3}+4C}} \frac{1}{\sqrt{\frac{k}{mr^3} + 4c}}$$

$$T_{prec} = \frac{4\pi(\sqrt{\frac{k}{mr^3} + 4C})}{\frac{k}{mr^3} + 4C - (\frac{k}{mr^3} + C)} = \frac{4\pi}{3C}\sqrt{\frac{k}{mr^3} + 4C}$$

Because C is very small compared to k, the approximation holds,

$$T_{prec} \approx \frac{4\pi}{3C}\sqrt{\frac{k}{mr^3}}$$

Therefore,

$$T_{prec} = \frac{4\pi}{3C}\omega_0 \quad \rightarrow \quad f_{prec} = \frac{3C}{4\pi\omega_0} \quad \rightarrow \quad \omega_{prec} = \frac{3C}{2\omega_0}$$

where $\omega_0 = \sqrt{\frac{k}{mr^3}}$.

Homework 5: # 3.31, 3.32, 3.7a

Michael Good

Sept 27, 2004

3.7a Show that the angle of recoil of the target particle relative to the incident direction of the scattered particle is simply $\Phi = \frac{1}{2}(\pi - \Theta)$.

Answer:

It helps to draw a figure for this problem. I don't yet know how to do this in LaTeX, but I do know that in the center of mass frame both the particles momentum are equal.

$$m_1 v_1' = m_2 v_2'$$

Where the prime indicates the CM frame. If you take equation (3.2) Goldstein, then its easy to understand the equation after (3.110) for the relationship of the relative speed v after the collision to the speed in the CM system.

$$v_1' = \frac{\mu}{m_1} v = \frac{m_2}{m_1 + m_2} v$$

Here, v is the relative speed *after* the collision, but as Goldstein mentions because elastic collisions conserve kinetic energy, (I'm assuming this collision is elastic even though it wasn't explicitly stated), we have $v = v_0$, that is the relative speed after collision is equal to the initial velocity of the first particle in the laboratory frame (the target particle being stationary).

$$v_1' = \frac{m_2}{m_1 + m_2} v_0$$

This equation works the same way for v_2'

$$v_2' = \frac{m_1}{m_1 + m_2} v_0$$

From conservation of momentum, we know that the total momentum in the CM frame is equal to the incident(and thus total) momentum in the laboratory frame.

$$(m_1 + m_2) v_{cm} = m_1 v_0$$

We see

$$v_{cm} = \frac{m_1}{m_1 + m_2} v_0$$

This is the same as v_2'

$$v_2' = v_{cm}$$

If we draw both frames in the same diagram, we can see an isosceles triangle where the two equal sides are v_2' and v_{cm}.

$$\Phi + \Phi + \Theta = \pi$$

$$\Phi = \frac{1}{2}(\pi - \Theta)$$

3.31 Examine the scattering produced by a repulsive central force $f + kr^{-3}$. Show that the differential cross section is given by

$$\sigma(\Theta) d\Theta = \frac{k}{2E} \frac{(1-x)dx}{x^2(2--x)^2 \sin \pi x}$$

where x is the ratio of Θ/π and E is the energy.

Answer:

The differential cross section is given by Goldstein (3.93):

$$\sigma(\Theta) = \frac{s}{\sin \Theta} \left| \frac{ds}{d\Theta} \right|$$

We must solve for s, and $ds/d\Theta$. Lets solve for $\Theta(s)$ first, take its derivative with respect to s, and invert it to find $ds/d\Theta$. We can solve for $\Theta(s)$ by using Goldstein (3.96):

$$\Theta(s) = \pi - 2 \int_{r_m}^{\infty} \frac{sdr}{r\sqrt{r^2(1 - \frac{V(r)}{E}) - s^2}}$$

What is $V(r)$ for our central force of $f = k/r^3$? It's found from $-dV/dr = f$.

$$V(r) = \frac{k}{2r^2}$$

Plug this in to Θ and we have

$$\Theta(s) = \pi - 2\int_{r_m}^{\infty} \frac{sdr}{r\sqrt{r^2 - (s^2 + \frac{k}{2E})}}$$

Before taking this integral, I'd like to put it in a better form. If we look at the energy of the incoming particle,

$$E = \frac{1}{2}mr_m^2\dot{\theta}^2 + \frac{k}{2r_m^2} = \frac{s^2E}{r_m^2} + \frac{k}{2r_m^2}$$

where from Goldstein page 113,

$$\dot{\theta}^2 = \frac{2s^2E}{mr_m^4}$$

We can solve for $s^2 + \frac{k}{2E}$, the term in Θ,

$$r_m^2 = s^2 + \frac{k}{2E}$$

Now we are in a better position to integrate,

$$\Theta(s) = \pi - 2\int_{r_m}^{\infty} \frac{sdr}{r\sqrt{r^2 - r_m^2}} = \pi - 2s\left[\frac{1}{r_m}\cos^{-1}\frac{r_m}{r}\Big|_{r_m}^{\infty}\right] = \pi - 2s\frac{1}{r_m}\left(\frac{\pi}{2}\right) = \pi\left(1 - \frac{s}{\sqrt{s^2 + \frac{k}{2E}}}\right)$$

Goldstein gave us $x = \Theta/\pi$, so now we have an expression for x in terms of s, lets solve for s

$$x = \frac{\Theta}{\pi} = 1 - \frac{s}{\sqrt{s^2 + \frac{k}{2E}}}$$

$$s^2 = (s^2 + \frac{k}{2E})(1-x)^2 \quad \rightarrow \quad s^2 = \frac{\frac{k}{2E}(1-x)^2}{1-(1-x)^2}$$

$$s = \sqrt{\frac{k}{2E}}\frac{(1-x)}{\sqrt{x(2-x)}}$$

Now that we have s we need only $ds/d\Theta$ to find the cross section. Solving $d\Theta/ds$ and then taking the inverse,

$$\frac{d\Theta}{ds} = \pi s\left(-\frac{1}{2}(s^2 + \frac{k}{2E})^{-\frac{3}{2}}\right)2s + \frac{\pi}{\sqrt{s^2 + \frac{k}{2E}}}$$

$$\frac{d\Theta}{ds} = \frac{-\pi s^2 + \pi(s^2 + \frac{k}{2E})}{(s^2 + \frac{k}{2E})^{\frac{3}{2}}} = \frac{\frac{\pi k}{2E}}{(s^2 + \frac{k}{2E})^{\frac{3}{2}}}$$

So

$$\frac{ds}{d\Theta} = \frac{2E(s^2 + \frac{k}{2E})^{\frac{3}{2}}}{\pi k}$$

Putting everything in terms of x,

$$s^2 + \frac{k}{2E} = \frac{k}{2E}\frac{(1-x)^2}{x(2-x)} + \frac{k}{2E} = \frac{k}{2E}\frac{1}{x(2-x)}$$

So now,

$$\sigma(\Theta) = \frac{s}{\sin\Theta}\left|\frac{ds}{d\Theta}\right| = \frac{\sqrt{\frac{k}{2E}}\frac{(1-x)}{\sqrt{x(2-x)}}}{\sin\pi x}\frac{2E(s^2+\frac{k}{2E})^{\frac{3}{2}}}{\pi k} = \frac{\sqrt{\frac{k}{2E}}\frac{(1-x)}{\sqrt{x(2-x)}}}{\sin\pi x}\frac{2E(\frac{k}{2E}\frac{1}{x(2-x)})^{\frac{3}{2}}}{\pi k} =$$

And this most beautiful expression becomes..

$$\sigma(\Theta) = \frac{1}{\sin\pi x}\frac{1}{\pi}(\frac{k}{2E})^{\frac{1}{2}}(\frac{2E}{k})(\frac{k}{2E})^{\frac{3}{2}}\frac{1-x}{\sqrt{x(2-x)}}\frac{1}{(x(2-x))^{\frac{3}{2}}}$$

After a bit more algebra...

$$\sigma(\Theta) = \frac{k}{2E}\frac{1}{\pi}\frac{1}{\sin\pi x}\frac{1-x}{(x(2-x))^2}$$

And since we know $d\Theta = \pi dx$,

$$\sigma(\Theta)d\Theta = \frac{k}{2E}\frac{(1-x)dx}{x^2(2-x)^2\sin\pi x}$$

3.32 A central force potential frequently encountered in nuclear physics is the rectangular well, defined by the potential

$$V = 0 \qquad r > a$$

$$V = -V_0 \qquad r \leq a$$

Show that the scattering produced by such a potential in classical mechanics is identical with the refraction of light rays by a sphere of radius a and relative index of refraction

$$n = \sqrt{\frac{E+V_0}{E}}$$

This equivalence demonstrates why it was possible to explain refraction phenomena both by Huygen's waves and by Newton's mechanical corpuscles. Show also that the differential cross section is

$$\sigma(\Theta) = \frac{n^2a^2}{4\cos\frac{\Theta}{2}}\frac{(n\cos\frac{\Theta}{2}-1)(n-\cos\frac{\Theta}{2})}{(1+n^2-2n\cos\frac{\Theta}{2})^2}$$

What is the total cross section?

Answer:

Ignoring the first part of the problem, and just solving for the differential cross section,

$$\sigma(\Theta) = \frac{sds}{\sin\Theta d\Theta}$$

If the scattering is the same as light refracted from a sphere, then putting our total angle scattered, Θ, in terms of the angle of incidence and transmission,

$$\Theta = 2(\theta_1 - \theta_2)$$

This is because the light is refracted from its horizontal direction twice, after hitting the sphere and leaving the sphere. Where $\theta_1 - \theta_2$ is the angle south of east for one refraction.

We know $\sin\theta_1 = s/a$ and using Snell's law, we know

$$n = \frac{\sin\theta_1}{\sin\theta_2} \to \sin\theta_2 = \frac{s}{na}$$

Expressing Θ in terms of just s and a we have

$$\Theta = 2(\arcsin\frac{s}{a} - \arcsin\frac{s}{na})$$

Now the plan is, to solve for s^2 and then $ds^2/d\Theta$ and solve for the cross section via

$$\sigma = \frac{sds}{\sin\Theta d\Theta} = \frac{1}{2\sin\Theta}\frac{ds^2}{d\Theta} = \frac{1}{4\sin\frac{\Theta}{2}\cos\frac{\Theta}{2}}\frac{ds^2}{d\Theta}$$

Here goes. Solve for $\sin\frac{\Theta}{2}$ and $\cos\frac{\Theta}{2}$ in terms of s

$$\sin\frac{\Theta}{2} = \sin(\arcsin\frac{s}{a} - \arcsin\frac{s}{na}) = \sin\arcsin\frac{s}{a}\cos\arcsin\frac{s}{na} - \cos\arcsin\frac{s}{a}\sin\arcsin\frac{s}{na}$$

This is

$$= \frac{s}{a}\cos(\arccos\sqrt{1-\frac{s^2}{n^2a^2}}) - \cos(\arccos(\sqrt{1-\frac{s^2}{a^2}})\frac{s}{na}$$

Using $\arcsin x = \arccos\sqrt{1-x^2}$ and $\sin(a-b) = \sin a\cos b - \cos a\sin b$. Now we have

$$\sin\frac{\Theta}{2} = \frac{s}{na^2}(\sqrt{n^2a^2-s^2}) - \sqrt{a^2-s^2})$$

Doing the same thing for $\cos\frac{\Theta}{2}$ yields

$$\cos\frac{\Theta}{2}=\frac{1}{na^2}(\sqrt{a^2-s^2}\sqrt{n^2a^2-s^2}+s^2)$$

Using $\cos(a-b)=\cos a\cos b+\sin a\sin b$. Still solving for s^2 in terms of cos and sin's we proceed

$$\sin^2\frac{\Theta}{2}=\frac{s^2}{n^2a^4}(n^2a^2-s^2-2\sqrt{n^2a^2-s^2}\sqrt{a^2-s^2}+a^2-s^2)$$

This is

$$\sin^2\frac{s^2}{n^2a^2}(n^2+1)-\frac{2s^4}{n^2a^4}-\frac{2s^2}{n^2a^4}\sqrt{n^2a^2-s^2}\sqrt{a^2-s^2}$$

Note that

$$\sqrt{n^2a^2-s^2}\sqrt{a^2-s^2}=na^2\cos\frac{\Theta}{2}-s^2$$

So we have

$$\sin^2\frac{\Theta}{2}=\frac{s^2}{n^2a^2}(n^2+1-\frac{2s^2}{a^2}-2n\cos\frac{\Theta}{2}+\frac{2s^2}{a^2})=\frac{s^2}{n^2a^2}(1+n^2-2n\cos\frac{\Theta}{2})$$

Solving for s^2

$$s^2=\frac{n^2a^2\sin^2\frac{\Theta}{2}}{1+n^2-2n\cos\frac{\Theta}{2}}$$

Glad that that mess is over with, we can now do some calculus. I'm going to let q^2 equal the denominator squared. Also to save space, lets say $\frac{\Theta}{2}=Q$. I like using the letter q.

$$\frac{ds^2}{d\Theta}=\frac{a^2\sin Qn^2}{q^2}[\cos Q(1-2n\cos Q+n^2)-n\sin^2 Q]$$

$$\frac{ds^2}{d\Theta}=\frac{n^2a^2}{q^2}\sin Q[\cos Q-2n\cos^2 Q+n^2\cos Q-n(1-\cos^2 Q)]$$

Expand and collect

$$\frac{ds^2}{d\Theta}=\frac{n^2a^2}{q^2}\sin Q[-n\cos^2 Q+\cos Q+n^2\cos Q-n]$$

Group it up

$$\frac{ds^2}{d\Theta}=\frac{n^2a^2}{q^2}\sin Q(n\cos Q-1)(n-\cos Q)$$

Plug back in for Q and q^2:

$$\frac{ds^2}{d\Theta} = \frac{n^2 a^2 \sin\frac{\Theta}{2}(n\cos\frac{\Theta}{2} - 1)(n - \cos\frac{\Theta}{2})}{(1 - 2n\cos\frac{\Theta}{2} + n^2)^2}$$

Using our plan from above,

$$\sigma = \frac{1}{4\sin\frac{\Theta}{2}\cos\frac{\Theta}{2}}\frac{ds^2}{d\Theta} = \frac{1}{4\sin\frac{\Theta}{2}\cos\frac{\Theta}{2}}\frac{n^2 a^2 \sin\frac{\Theta}{2}(n\cos\frac{\Theta}{2} - 1)(n - \cos\frac{\Theta}{2})}{(1 - 2n\cos\frac{\Theta}{2} + n^2)^2}$$

We obtain

$$\sigma(\Theta) = \frac{1}{4\cos\frac{\Theta}{2}}\frac{n^2 a^2(n\cos\frac{\Theta}{2} - 1)(n - \cos\frac{\Theta}{2})}{(1 - 2n\cos\frac{\Theta}{2} + n^2)^2}$$

The total cross section involves an algebraic intensive integral. The total cross section is given by

$$\sigma_T = 2\pi\int_0^{\Theta_{max}} \sigma(\Theta)\sin\Theta d\Theta$$

To find Θ_{max} we look for when the cross section becomes zero. When $(n\cos\frac{\Theta}{2} - 1)$ is zero, we'll have Θ_{max}. If $s > a$, its as if the incoming particle misses the 'sphere'. At $s = a$ we have maximum Θ. So using $\Theta_{max} = 2\arccos\frac{1}{n}$, we will find it easier to plug in $x = \cos\frac{\Theta}{2}$ as a substitution, to simplify our integral.

$$\sigma_T = \pi\int_{\frac{1}{n}}^{1} a^2 n^2 \frac{(nx-1)(n-x)}{(1-2nx+n^2)^2}2dx$$

where

$$dx = -\frac{1}{2}\sin\frac{\Theta}{2}d\Theta \qquad \cos\frac{\Theta_{max}}{2} = \frac{1}{n}$$

The half angle formula, $\sin\Theta = 2\sin\frac{\Theta}{2}\cos\frac{\Theta}{2}$ was used on the $\sin\Theta$, the negative sign switched the direction of integration, and the factor of 2 had to be thrown in to make the dx substitution.

This integral is still hard to manage, so make another substitution, this time, let q equal the term in the denominator.

$$q = 1 - 2nx + n^2 \qquad \rightarrow \qquad dq = -2ndx$$

The algebra must be done carefully here. Making a partial substitution to see where to go:

$$q_{min} = 1 - 2 + n^2 = n^2 - 1 \qquad q_{max} = n^2 - 2n + 1 = (n-1)^2$$

$$\sigma_T = \int_{n^2-1}^{(n-1)^2} \frac{2\pi a^2 n^2(nx-1)(n-x)}{q^2}\frac{dq}{-2n} = \pi a^2\int_{n^2-1}^{(n-1)^2}\frac{-n(nx-1)(n-x)}{q^2}dq$$

Expanding q^2 to see what it gives so we can put the numerator in the above integral in terms of q^2 we see

$$q^2 = n^4 + 1 + 2n^2 - 4n^3x - 4nx + 4n^2x^2$$

Expanding the numerator

$$-n(nx-1)(n-x) = -n^3x - nx + n^2x^2 + n^2$$

If we take q^2 and subtract a n^4, subtract a 1, add a $2n^2$ and divide the whole thing by 4 we'll get the above numerator. That is:

$$\frac{q^2 - n^4 + 2n^2 - 1}{4} = \frac{q^2 - (n^2-1)^2}{4} = -n(nx-1)(n-x)$$

Now, our integral is

$$\sigma_T = \pi a^2 \int_{n^2-1}^{(n-1)^2} \frac{q^2 - (n^2-1)^2}{4q^2} dq$$

This is finally an integral that can be done by hand

$$\sigma_T = \frac{\pi a^2}{4} \int 1 - \frac{(n^2-1)^2}{q^2} dq = \frac{\pi a^2}{4} \left(z + \frac{(n^2-1)^2}{z}\Big|_{n^2-1}^{(n-1)^2}\right)$$

After working out the few steps of algebra,

$$\frac{\pi a^2}{4} \frac{4n^2 - 8n + 4}{n^2 - 2n + 1} = \pi a^2$$

The total cross section is

$$\sigma_T = \pi a^2$$

Homework 6: # 4.1, 4.2, 4.10, 4.14, 4.15

Michael Good

Oct 4, 2004

4.1
Prove that matrix multiplication is associative. Show that the product of two orthogonal matrices is also orthogonal.

Answer:

Matrix associativity means

$$A(BC) = (AB)C$$

The elements for any row i and column j, are

$$A(BC) = \sum_k A_{ik}(\sum_m B_{km}C_{mj})$$

$$(AB)C = \sum_m(\sum_k A_{ik}B_{km})C_{mj}$$

Both the elements are the same. They only differ in the order of addition. As long as the products are defined, and there are finite dimensions, matrix multiplication is associative.

Orthogonality may be defined by

$$\widetilde{A}A = I$$

The Pauli spin matrices, σ_x, and σ_z are both orthogonal.

$$\tilde{\sigma_x}\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$$\tilde{\sigma_z}\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

The product of these two:

$$\sigma_x\sigma_z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \equiv q$$

is also orthogonal:

$$\tilde{q}q = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

More generally, if

$$\widetilde{A}A = 1 \qquad \widetilde{B}B = 1$$

then both A, and B are orthogonal. We can look at

$$\widetilde{AB}AB = \sum_k (\widetilde{AB})_{ik}(AB)_{kj} = \sum_k AB_{ki}AB_{kj} = \sum_{k,s,r} a_{ks}b_{si}a_{kr}b_{rj}$$

The elements are

$$\sum_{k,s,r} a_{ks}b_{si}a_{kr}b_{rj} = \sum b_{si}a_{ks}a_{kr}b_{rj} = \sum b_{si}(\tilde{A}A)_{sr}b_{rj}$$

This is

$$\widetilde{AB}AB = \sum b_{si}\delta_{sr}b_{rj} = \tilde{B}B_{ij} = \delta_{ij}$$

Therefore the whole matrix is I and the product

$$\widetilde{AB}AB = I$$

is orthogonal.

4.2
Prove the following properties of the transposed and adjoint matrices:

$$\widetilde{AB} = \widetilde{B}\widetilde{A}$$

$$(AB)^\dagger = B^\dagger A^\dagger$$

Answer:

For transposed matrices

$$\widetilde{AB} = \widetilde{AB}_{ij} = AB_{ji} = \sum a_{js}b_{si} = \sum b_{si}a_{js} = \sum \widetilde{B}_{is}\widetilde{A}_{sj} = (\widetilde{B}\widetilde{A})_{ij} = \widetilde{B}\widetilde{A}$$

As for the complex conjugate,

$$(AB)^\dagger = (\widetilde{AB})^*$$

From our above answer for transposed matrices we can say

$$\widetilde{AB} = \widetilde{B}\widetilde{A}$$

And so we have

$$(AB)^\dagger = (\widetilde{AB})^* = (\widetilde{B}\widetilde{A})^* = \widetilde{B}^*\widetilde{A}^* = B^\dagger A^\dagger$$

4.10

If B is a square matrix and A is the exponential of B, defined by the infinite series expansion of the exponential,

$$A \equiv e^B = 1 + B + \frac{1}{2}B^2 + ... + \frac{B^n}{n!} + ...,$$

then prove the following properties:

- $e^B e^C = e^{B+C}$, providing B and C commute.
- $A^{-1} = e^{-B}$
- $e^{CBC^{-1}} = CAC^{-1}$
- A is orthogonal if B is antisymmetric

Answer:

Providing that B and C commute;

$$BC - CB = 0 \qquad BC = CB$$

we can get an idea of what happens:

$$(1+B+\frac{B^2}{2}+O(B^3))(1+C+\frac{C^2}{2}+O(C^3)) = 1+C+\frac{C^2}{2}+B+BC+\frac{B^2}{2}+O(3)$$

This is

$$1+(B+C)+\frac{1}{2}(C^2+2BC+B^2)+O(3) = 1+(B+C)+\frac{(B+C)^2}{2}+O(3) = e^{B+C}$$

Because $BC = CB$ and where $O(3)$ are higher order terms with products of 3 or more matrices. Looking at the kth order terms, we can provide a rigorous proof.

Expanding the left hand side of

$$e^B e^C = e^{B+C}$$

and looking at the kth order term, by using the expansion for exp we get, noting that $i + j = k$

$$\sum_0^k \frac{B^i C^j}{i!j!} = \sum_0^k \frac{B^{k-j}C^j}{(k-j)!j!}$$

and using the binomial expansion on the right hand side for the kth order term, (a proof of which is given in Riley, Hobsen, Bence):

$$\frac{(B+C)^k}{k!} = \frac{1}{k!}\sum_0^k \frac{k!}{(k-j)!j!}B^{k-j}C^j = \sum_0^k \frac{B^{k-j}C^j}{(k-j)!j!}$$

we get the same term. QED.
To prove

$$A^{-1} = e^{-B}$$

We remember that

$$A^{-1}A = 1$$

and throw e^{-B} on the right

$$A^{-1}Ae^{-B} = 1e^{-B}$$

$$A^{-1}e^{B}e^{-B} = e^{-B}$$

and from our above proof we know $e^B e^C = e^{B+C}$ so

$$A^{-1}e^{B-B} = e^{-B}$$

Presto,

$$A^{-1} = e^{-B}$$

To prove

$$e^{CBC^{-1}} = CAC^{-1}$$

its best to expand the *exp*

$$\sum_0^\infty \frac{1}{n!}(CBC^{-1})^n = 1+CBC^{-1}+\frac{CBC^{-1}CBC^{-1}}{2}+....+\frac{CBC^{-1}CBC^{-1}CBC^{-1}...}{n!}+...$$

Do you see how the middle $C^{-1}C$ terms cancel out? And how they cancel each out n times? So we are left with just the C and C^{-1} on the outside of the B's.

$$\sum_0^\infty \frac{1}{n!}(CBC^{-1})^n = \sum_0^\infty \frac{1}{n!}CB^nC^{-1} = Ce^BC^{-1}$$

Remember $A = e^B$ and we therefore have

$$e^{CBC^{-1}} = CAC^{-1}$$

To prove A is orthogonal

$$\widetilde{A} = A^{-1}$$

if B is antisymmetric

$$-B = \widetilde{B}$$

We can look at the transpose of A

$$\widetilde{A} = \widetilde{\sum_0^\infty \frac{B^n}{n!}} = \sum_0^\infty \frac{\widetilde{B}^n}{n!} = \sum_0^\infty \frac{(-B)^n}{n!} = e^{-B}$$

But from our second proof, we know that $e^{-B} = A^{-1}$, so

$$\widetilde{A} = A^{-1}$$

and we can happily say A is orthogonal.

4.14

- Verify that the permutation symbol satisfies the following identity in terms of Kronecker delta symbols:

$$\epsilon_{ijp}\epsilon_{rmp} = \delta_{ir}\delta_{jm} - \delta_{im}\delta_{jr}$$

- Show that

$$\epsilon_{ijp}\epsilon_{ijk} = 2\delta_{pk}$$

Answer:
To verify this first identity, all we have to do is look at the two sides of the equation, analyzing the possibilities, i.e. if the right hand side has

$$i = r \quad j = m \neq i$$

we get $+1$. If

$$i = m \quad j = r \neq i$$

we get -1. For any other set of i, j, r, and m we get 0.

For the left hand side, lets match conditions, if

$$i = r \quad j = m \neq i$$

then $\epsilon_{ijp} = \epsilon_{rmp}$ and whether or not ϵ_{ijp} is ± 1 the product of the two gives a $+1$. If

$$i = m \quad j = r \neq i$$

then $\epsilon_{rmp} = \epsilon_{jip} = -\epsilon_{ijp}$ and whether or not ϵ_{ijp} is ± 1 the product is now equal to -1.

These are the only nonzero values because for i, j, r, m, none can have the same value as p. Since there are only three values, that any of the subscripts may take, the only non-zero values are the ones above. (not all four subscripts may be equal because then it would be $\epsilon = 0$ as if $i = j$ or $r = m$).

To show that

$$\epsilon_{ijp}\epsilon_{ijk} = 2\delta_{pk}$$

we can use our previous identity, cast in a different form:

$$\epsilon_{ijk}\epsilon_{imp} = \delta_{jm}\delta_{kp} - \delta_{jp}\delta_{km}$$

This is equivalent because the product of two Levi-Civita symbols is found from the deteriment of a matrix of delta's, that is

$$\epsilon_{ijk}\epsilon_{rmp} = \delta_{ir}\delta_{jm}\delta_{kp} + \delta_{im}\delta_{jp}\delta_{kr} + \delta_{ip}\delta_{jr}\delta_{km} - \delta_{im}\delta_{jr}\delta_{kp} - \delta_{ir}\delta_{jp}\delta_{km} - \delta_{ip}\delta_{jm}\delta_{kr}$$

For our different form, we set $i = r$. If we also set $j = m$, this is called 'contracting' we get

$$\epsilon_{ijk}\epsilon_{ijp} = \delta_{jj}\delta_{kp} - \delta_{jp}\delta_{kj}$$

Using the summation convention, $\delta_{jj} = 3$,

$$\epsilon_{ijk}\epsilon_{ijp} = 3\delta_{kp} - \delta_{kp}$$

$$\epsilon_{ijk}\epsilon_{ijp} = 2\delta_{kp}$$

4.15
Show that the components of the angular velocity along the space set of axes are given in terms of the Euler angles by

$$\omega_x = \dot{\theta}\cos\phi + \dot{\psi}\sin\theta\sin\phi$$
$$\omega_y = \dot{\theta}\sin\phi - \dot{\psi}\sin\theta\cos\phi$$
$$\omega_z = \dot{\psi}\cos\theta + \dot{\phi}$$

Answer:

Using the same analysis that Goldstein gives to find the angular velocity along the body axes (x', y', z') we can find the angular velocity along the space axes (x, y, z). To make a drawing easier, its helpful to label the axes of rotation for $\dot{\theta}$, $\dot{\psi}$ and $\dot{\phi}$.

$$\dot{\theta} \to L.O.N.$$
$$\dot{\psi} \to z'$$
$$\dot{\phi} \to z$$

We want

$$\omega_x = \dot{\theta}_x + \dot{\psi}_x + \dot{\phi}_x$$
$$\omega_y = \dot{\theta}_y + \dot{\psi}_y + \dot{\phi}_y$$
$$\omega_z = \dot{\theta}_z + \dot{\psi}_z + \dot{\phi}_z$$

Lets start with ω_z first to be different. If we look at the diagram carefully on page 152, we can see that $\dot{\theta}$ is along the line of nodes, that is θ revolves around the line of nodes. Therefore because the line of nodes is perpendicular to the z space axis there is no component of θ contributing to angular velocity around the z space axis. $\dot{\theta}_z = 0$. What about $\dot{\psi}_z$? Well, ψ revolves around z'. So there is a component along z due to a changing ψ. That component depends on how much angle there is between z' and z, which is θ. Does this makes sense? We find the z part, which is the adjacent side to θ. Thus we have $\dot{\psi}_z = \dot{\psi}\cos\theta$. Now lets look at $\dot{\phi}_z$. We can see that ϕ just revolves around z in the first place! Right? So there is no need to make any 'transformation' or make any changes. Lets take $\dot{\phi}_z = \dot{\phi}$. Add them all up for our total ω_z.

$$\omega_z = \dot{\theta}_z + \dot{\psi}_z + \dot{\phi}_z = 0 + \dot{\psi}\cos\theta + \dot{\phi}$$

Now lets do the harder ones. Try ω_x. What is $\dot{\theta}_x$? Well, $\dot{\theta}$ is along the line of nodes, that is, θ changes and revolves around the line of nodes axis. To find the x component of that, we just see that the angle between the line of nodes and the x axis is only ϕ, because they both lie in the same xy plane. Yes? So $\dot{\theta}_x = \dot{\theta}\cos\phi$. The adjacent side to ϕ with $\dot{\theta}$ as the hypotenuse. Lets look at $\dot{\phi}_x$. See how ϕ revolves around the z axis? Well, the z axis is perpendicular to the x axis there for there is no component of $\dot{\phi}$ that contributes to the x space axis. $\dot{\phi}_x = 0$. Now look at $\dot{\psi}_x$. We can see that $\dot{\psi}$ is along the z' body axis, that is, it is in a whole different plane than x. We first have to find the component in the same xy plane, then find the component of the x direction. So to get into the xy plane we can take $\dot{\psi}_{x,y} = \dot{\psi}\sin\theta$. Now its in the same plane. But where is it facing in this plane? We can see that depends on the angle ϕ. If $\phi = 0$ we would have projected it right on top of the $y - axis$! So we can make sure that if $\phi = 0$ we have a zero component for x by multiplying by $\sin\phi$. So we get after two projections, $\dot{\psi}_x = \dot{\psi}\sin\theta\sin\phi$. Add these all up for our total ω_x, angular velocity in the x space axis.

$$\omega_x = \dot{\theta}_x + \dot{\psi}_x + \dot{\phi}_x = \dot{\theta}\cos\phi + \dot{\psi}\sin\theta\sin\phi + 0$$

I'll explain ω_y for kicks, even though the process is exactly the same. Look for $\dot{\theta}_y$. $\dot{\theta}$ is along the line of nodes. Its y component depends on the angle ϕ. So project it to the y axis. $\dot{\theta}_y = \dot{\theta}\sin\phi$. Look for $\dot{\psi}_y$. Its in a different plane again, so two projections are necessary to find its component. Project down to the xy plane like we did before, $\dot{\psi}_{x,y} = \dot{\psi}\sin\theta$ and now we remember that if $\phi = 0$ we would have exactly placed it on top of the y axis. Thats good! So lets make it if $\phi = 0$ we have the full $\dot{\psi}\sin\theta$, (ie multiply by $\cos\phi$ because $\cos 0 = 1$). But we also have projected it in the opposite direction of the positive y direction, (throw in a negative). So we have $\dot{\psi}_y = -\dot{\psi}\sin\theta\cos\phi$. For $\dot{\phi}_y$ we note that ϕ revolves around the z axis, completely perpendicular to y. Therefore no component in the y direction. $\dot{\phi}_y = 0$. Add them all up

$$\omega_y = \dot{\theta}_y + \dot{\psi}_y + \dot{\phi}_y = \dot{\theta}\sin\phi - \dot{\psi}\sin\theta\cos\phi + 0$$

Here is all the ω's together

$$\omega_x = \dot{\theta}\cos\phi + \dot{\psi}\sin\theta\sin\phi$$
$$\omega_y = \dot{\theta}\sin\phi - \dot{\psi}\sin\theta\cos\phi$$
$$\omega_z = \dot{\psi}\cos\theta + \dot{\phi}$$

Homework 7: # 4.22, 5.15, 5.21, 5.23, Foucault pendulum

Michael Good

Oct 9, 2004

4.22
A projectile is fired horizontally along Earth's surface. Show that to a first approximation the angular deviation from the direction of fire resulting from the Coriolis effect varies linearly with time at a rate

$$\omega \cos\theta$$

where ω is the angular frequency of Earth's rotation and θ is the co-latitude, the direction of deviation being to the right in the northern hemisphere.

Answer:

I'll call the angular deviation ψ. We are to find

$$\psi = \omega \cos\theta t$$

We know ω is directed north along the axis of rotation, that is, sticking out of the north pole of the earth. We know θ is the co-latitude, that is, the angle from the poles to the point located on the surface of the Earth. The latitude, λ is the angle from the equator to the point located on the surface of the Earth. $\lambda = \pi/2 - \theta$. Place ourselves in the coordinate system of whoever may be firing the projectile on the surface of the Earth. Call y' the horizontal direction pointing north (not toward the north pole or into the ground, but horizontally north), call x' the horizontal direction pointed east, and call z' the vertical direction pointed toward the sky.

With our coordinate system in hand, lets see where ω is. Parallel transport it to the surface and note that it is between y' and z'. If we are at the north pole, it is completely aligned with z', if we are at the equator, ω is aligned with y'. Note that the angle between z' and ω is the co-latitude, θ.(θ is zero at the north pole, when ω and z' are aligned). If we look at the components of ω, we can take a hint from Goldstein's Figure 4.13, that deflection of the horizontal trajectory in the northern hemisphere will depend on only the z' component of ω, labeled $\omega_{z'}$. Only ω_z is used for our approximation. It is clear that there is no component of ω in the x' direction. If we took into account the component in the y' direction we would have an effect causing the particle to move into the vertical direction, because the Coriolis effect is

$$F_c = -2m(\omega \times v)$$

and $\omega_y \times v$ would add a contribution in the z direction because our projectile is fired only along x' and y', that is, horizontally. So following Goldstein's figure, we shall only be concerned with ω_z. Our acceleration due to the Coriolis force is

$$a_c = -2(\omega \times v) = 2(v \times \omega)$$

The component of ω in the z' direction is $\omega_{z'} = \omega \cos\theta$. Thus the magnitude of the acceleration is

$$a_c = 2v\omega\cos\theta$$

The distance affected by this acceleration can be found through the equation of motion,

$$d = \frac{1}{2}a_c t^2 = v\omega\cos\theta t^2$$

And using a small angle of deviation, for ψ we can draw a triangle and note that the distance traveled by the projectile is just $x = vt$.

$$x\psi = d \quad \rightarrow \quad \psi = \frac{d}{x}$$

$$\psi = \frac{v\omega\cos\theta t^2}{vt} = \omega\cos\theta t$$

Therefore the angular deviation varies linearly on time with a rate of $\omega\cos\theta$. Note that there is no Coriolis effect at the equator when $\theta = \pi/2$, therefore no angular deviation.

5.15
Find the principal moments of inertia about the center of mass of a flat rigid body in the shape of a 45^o right triangle with uniform mass density. What are the principal axes?

Answer:

Using the moment of inertia formula for a lamina, which is a flat closed surface, (as explained on wolfram research) we can calculate the moment of inertia for the triangle, with it situated with long side on the x-axis, while the y-axis cuts through the middle. The off-diagonal elements of the inertia tensor vanish.

$$I_x = \int \sigma y^2 dxdy = 2\int_0^a \int_0^{a-x} \frac{M}{A} y^2 dydx = \frac{2M}{a^2}\int_0^a \frac{(a-x)^3}{3} dx$$

Solving the algebra,

$$I_x = \frac{2M}{3a^2}\int_0^a (-x^3 + 3ax^2 - 3a^2x + a^3)dx = \frac{2Ma^2}{3}[\frac{8}{4} - \frac{1}{4} - \frac{6}{4}] = \frac{Ma^2}{6}$$

For I_y

$$I_y = \int \sigma x^2 dxdy = 2\int_0^a \int_0^{a-y} \frac{M}{A} x^2 dxdy$$

This has the exact same form, so if you're clever, you won't do the integral over again.

$$I_y = \frac{Ma^2}{6}$$

For I_z

$$I_z = \int \sigma(x^2 + y^2)dxdy = I_x + I_y = (\frac{1}{6} + \frac{1}{6})Ma^2 = \frac{Ma^2}{3}$$

We can use the parallel axis theorem to find the principal moments of inertia about the center of mass. The center of mass is

$$y_{cm} = 2\frac{\sigma}{M}\int_0^a \int_0^{a-x} ydxdy = \frac{2}{a^2}\int_0^a \frac{(a-x)^2}{2} dx$$

$$y_{cm} = \frac{1}{a^2}\int_0^a (a^2 - 2xa + x^2)dx = a^2x - ax^2 + \frac{x^3}{3}\bigg|_0^a \frac{1}{a^2} = \frac{a}{3}$$

From symmetry we can tell that the center of mass is $(0, \frac{a}{3}, 0)$. Using the parallel axis theorem, with $r_0 = a/3$

$$I_X = I_x - Mr_0^2$$

$$I_Y = I_y$$

$$I_Z = I_z - Mr_0^2$$

These are

$$I_X = (\frac{1}{6} - \frac{1}{9})Ma^2 = (\frac{3}{18} - \frac{2}{18})Ma^2 = \frac{Ma^2}{18}$$

$$I_Y = \frac{Ma^2}{6}$$

$$I_Z = (\frac{1}{3} - \frac{1}{9})Ma^2 = \frac{2}{9}Ma^2$$

5.21
A compound pendulum consists of a rigid body in the shape of a lamina suspended in the vertical plane at a point other than the center of gravity. Compute the period for small oscillations in terms of the radius of gyration about the center of gravity and the separation of the point of suspension from the center of gravity. Show that if the pendulum has the same period for two points of suspension at unequal distances from the center of gravity, then the sum of these distances is equal to the length of the equivalent simple pendulum.

Answer:

Looking for an equation of motion, we may equate the torque to the moment of inertia times the angular acceleration.

$$lF = I\ddot{\theta}$$

The force is $-Mg\sin\theta$, and the moment of inertia, using the parallel axis theorem is

$$I = Mr_g^2 + Ml^2$$

where r_g radius of gyration about the center of gravity, and l is the distance between the pivot point and center of gravity. The equation of motion becomes

$$-lMg\sin\theta = (Mr_g^2 + Ml^2)\ddot{\theta}$$

Using small oscillations, we can apply the small angle approximation $\sin\theta \approx \theta$

$$-lg\theta = (r_g^2 + l^2)\ddot{\theta}$$

$$\frac{lg}{r_g^2 + l^2}\theta + \ddot{\theta} = 0$$

This is with angular frequency and period

$$\omega = \sqrt{\frac{lg}{r_g^2 + l^2}} \quad \rightarrow \quad T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{r_g^2 + l^2}{lg}}$$

This is the same as the period for a physical pendulum

$$T = 2\pi\sqrt{\frac{I}{Mgl}} = 2\pi\sqrt{\frac{r_g^2 + l^2}{lg}}$$

If we have two points of suspension, l_1 and l_2, each having the same period, T. Then we get

$$2\pi\sqrt{\frac{r_g^2+l_1^2}{l_1 g}} = 2\pi\sqrt{\frac{r_g^2+l_2^2}{l_2 g}}$$

This is

$$\frac{r_g^2+l_1^2}{l_1} = \frac{r_g^2+l_2^2}{l_2}$$

And in a more favorable form, add l_1 to both sides, because we are looking for l_1+l_2 to be equivalent to a simple pendulum length,

$$\frac{r_g^2}{l_1}+l_1+l_1 = \frac{r_g^2}{l_2}+l_2+l_1$$

$$\frac{r_g^2}{l_1 l_2}(l_2-l_1)+2l_1 = l_2+l_1$$

This is only true if

$$r_g^2 = l_1 l_2$$

Thus our period becomes

$$T = 2\pi\sqrt{\frac{r_g^2+l_1^2}{l_1 g}} = 2\pi\sqrt{\frac{l_1 l_2+l_1^2}{l_1 g}} = 2\pi\sqrt{\frac{l_2+l_1}{g}} = 2\pi\sqrt{\frac{L}{g}}$$

where L is the length of a simple pendulum equivalent.

5.23
An automobile is started from rest with one of its doors initially at right angles. If the hinges of the door are toward the front of the car, the door will slam shut as the automobile picks up speed. Obtain a formula for the time needed for the door to close if the acceleration f is constant, the radius of gyration of the door about the axis of rotation is r_0 and the center of mass is at a distance a from the hinges. Show that if f is $0.3m/s^2$ and the door is a uniform rectangle is 1.2m wide, the time will be approximately 3.04 s.

Answer:

Begin by setting the torque equal to the product of the moment of inertia and angular acceleration.

$$I\ddot{\theta} = aF$$

The moment of inertia is $I = mr_0^2$. The force is $F = -mf\sin\theta$. So we get

$$mr_0^2\ddot{\theta} = -amf\sin\theta$$

Our equation of motion is

$$\ddot{\theta} = -\frac{af}{r_0^2}\sin\theta$$

This is rough. In our case we can not use the small angle approximation. The door starts at 90^o! How do we go about solving this then? Lets try integrating it once and see how far we can get. Here is a handy trick,

$$\ddot{\theta} = \frac{d}{dt}\frac{d\theta}{dt} = \frac{d\dot{\theta}}{dt} = \frac{d\dot{\theta}}{d\theta}\frac{d\theta}{dt} = \frac{d\dot{\theta}}{d\theta}\dot{\theta}$$

Plug this into our equation of motion

$$\frac{d\dot{\theta}}{d\theta}\dot{\theta} = -\frac{af}{r_0^2}\sin\theta$$

This is separable, and may be integrated.

$$\frac{\dot{\theta}^2}{2} = \frac{af}{r_0^2}\cos\theta$$

$$\dot{\theta} = \sqrt{\frac{2af}{r_0^2}\cos\theta}$$

The time may be found by integrating over the time of travel it takes for the door to shut.

$$T = \int_0^{\frac{\pi}{2}} \frac{dt}{d\theta}d\theta = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\dot{\theta}} = \int_0^{\frac{\pi}{2}} \sqrt{\frac{r_0^2}{2af}}\frac{d\theta}{\sqrt{\cos\theta}}$$

Here is where the physics takes a backseat for a few, while the math runs the show. If we throw in a $-\cos 90^o$ we might notice that this integral is an elliptic integral of the first kind, denoted K.

$$T = \sqrt{\frac{r_0^2}{2af}}\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{\cos\theta}} = \sqrt{\frac{r_0^2}{2af}}\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{\cos\theta - \cos\frac{\pi}{2}}} = \sqrt{\frac{r_0^2}{2af}}\sqrt{2}K(\sin\frac{\pi}{4})$$

This can be seen from mathworld's treatment of elliptic integrals, at http://mathworld.wolfram.com/EllipticIntegraloftheFirstKind.html.
Now we have

$$T = \sqrt{\frac{r_0^2}{af}}K(\frac{\sqrt{2}}{2})$$

$K(\frac{\sqrt{2}}{2})$ belongs to a group of functions called 'elliptic integral singular values', $K(k_r)$ A treatment of them and a table of their values that correspond to gamma functions are given here:
http://mathworld.wolfram.com/EllipticIntegralSingularValue.html.

The ‘elliptic lambda function’ determines the value of k_r. A table of lambda functions is here

http://mathworld.wolfram.com/EllipticLambdaFunction.html.

Our k_r value of $\frac{\sqrt{2}}{2}$ corresponds to k_1. From the singular value table,

$$K(k_1) = \frac{\Gamma^2(\frac{1}{4})}{4\sqrt{\pi}}$$

Our time is now

$$T = \sqrt{\frac{r_0^2}{af}}\frac{\Gamma^2(\frac{1}{4})}{4\sqrt{\pi}}$$

Fortunately, there are nice calculators that will compute gamma functions quickly. I used this one

http://www.efunda.com/math/gamma/findgamma.cfm.

I now have

$$\Gamma(\frac{1}{4}) = 3.63$$

Back to the physics. The moment of inertia of a uniform rectangle about the axis that bisects it is $\frac{M}{3}a^2$. Move the axis to the edge of the rectangle using the parallel axis theorem.

$$I = Mr_0^2 = Ma^2 + \frac{M}{3}a^2 = \frac{4}{3}Ma^2$$

we now have

$$r_0^2 = \frac{4}{3}a^2$$

With $a = .6m$, that is, half of the length of the car door, assuming its mass is uniform. And with $f = .3m/s^2$ we have

$$T = \sqrt{\frac{4a}{3f}}\frac{1}{4\sqrt{\pi}}(3.63)^2 = \sqrt{\frac{4(.6)}{3(.3)}}\frac{1}{4\sqrt{3.14}}(3.63)^2 = 3.035 \approx 3.04\ s$$

Foucault Pendulum
Find the period of rotation as a function of latitude.
Hint: neglect centrifugal force, neglect change in height, solve for $\xi = x + iy$

Answer:

The Foucault pendulum is a swinging weight supported by a long wire, so that the wire’s upper support restrains the wire only in the vertical direction and the weight is set swinging with no lateral or circular motion. The plane of the pendulum gradually rotates, demonstrating the Earth’s rotation. Solve for the period of rotation of this plane. The equation of motion for acceleration takes into account the vertical acceleration due to gravity, the acceleration from the tension and the Coriolis acceleration.

$$a_r = g + \frac{T}{m} - 2\omega \times v_r$$

In my system, I have x facing east, y facing north, and z facing to the sky. This yeilds

$$\omega_x = 0$$
$$\omega_y = \omega \sin\theta = \omega \cos\lambda$$
$$\omega_z = \omega \cos\theta = \omega \sin\lambda$$

The only velocity contributions come from the x and y components, for we can ignore the change in height. The Coriolis acceleration is quickly derived

$$a_c = \dot{y}\omega \sin\lambda \hat{x} - \dot{x}\omega \sin\lambda \hat{y} + \dot{x}\omega \cos\lambda \hat{z}$$

Looking for the period of rotation, we are concerned only with the x and y accelerations. Our overall acceleration equations become

$$\ddot{x} = -\frac{g}{l}x + 2\dot{y}\omega \sin\lambda$$
$$\ddot{y} = -\frac{g}{l}y - 2\dot{x}\omega \sin\lambda$$

The g/l terms were found using approximations for the tension components, that is, $T_x = -T\frac{x}{l} \rightarrow T_x/ml = g/l$ and the same for y.

Introducing $\xi = x + iy$ and adding the two equations after multiplying the second one by i

$$\ddot{\xi} + \frac{g}{l}\xi = -2\omega \sin\lambda(-\dot{y} + i\dot{x})$$
$$\ddot{\xi} + \frac{g}{l}\xi = -2i\omega \sin\lambda\dot{\xi}$$

$$\ddot{\xi} + \frac{g}{l}\xi + 2i\omega \sin\lambda\dot{\xi} = 0$$

This is the damped oscillation expression. It's solution is, using $\frac{g}{l} >> \omega \sin\lambda$, the over damped case

$$\xi = e^{-i\omega \sin\lambda t}(Ae^{i\sqrt{\frac{g}{l}}t} + Be^{-i\sqrt{\frac{g}{l}}t})$$

The equation for oscillation of a pendulum is

$$\ddot{q} + \frac{g}{l}q = 0$$

It has solution

$$q = Ae^{i\sqrt{\frac{g}{l}}t} + Be^{-i\sqrt{\frac{g}{l}}t}$$

We can simplify our expression then, using q

$$\xi = qe^{-i\omega \sin \lambda t}$$

Where the angular frequency of the plane's rotation is $\omega \cos \theta$, or $\omega \sin \lambda$ where λ is the latitude, and θ is the co-latitude. The period can be found using, $\omega = 2\pi/T$.

$$\frac{2\pi}{T_{earth}} \cos \theta = \frac{2\pi}{T_{Foucault}} \rightarrow T_{Foucault} = \frac{T_{Earth}}{\cos \theta}$$

This can be checked because we know the pendulum rotates completely in 1 day at the North pole where $\theta = 0$ and has no rotation at the equator where $\theta = 90^0$. Chapel Hill has a latitude of 36^o, a Foucault pendulum takes

$$T_{Foucault} = \frac{24\ hours}{\sin 36^o} \approx 41\ hours$$

to make a full revolution.

Homework 9: # 8.19, 8.24, 8.25

Michael Good

Nov 2, 2004

8.19
The point of suspension of a simple pendulum of length l and mass m is constrained to move on a parabola $z = ax^2$ in the vertical plane. Derive a Hamiltonian governing the motion of the pendulum and its point of suspension. Obtain the Hamilton's equations of motion.

Answer:

Let

$$x' = x + l\sin\theta$$
$$z' = ax^2 - l\cos\theta$$

Then

$$T = \frac{1}{2}m(\dot{x}'^2 + \dot{z}'^2)$$
$$U = mgz'$$

Solving in terms of generalized coordinates, x and θ, our Lagrangian is

$$L = T - U = \frac{1}{2}m(\dot{x}^2 + 2\dot{x}l\cos\theta\dot{\theta} + 4a^2x^2\dot{x}^2 + 4ax\dot{x}l\dot{\theta}\sin\theta + l^2\dot{\theta}^2) - mg(ax^2 - l\cos\theta)$$

Using

$$L = L_0 + \frac{1}{2}\tilde{\dot{q}}T\dot{q}$$

where $\dot{q}$ and T are matrices. We can see

$$\dot{q} = \begin{pmatrix} \dot{x} \\ \dot{\theta} \end{pmatrix}$$

$$T = \begin{pmatrix} m(1 + 4a^2x^2) & ml(\cos\theta + 2ax\sin\theta) \\ ml(\cos\theta + 2ax\sin\theta) & ml^2 \end{pmatrix}$$

with

$$L_0 = -mg(ax^2 - l\cos\theta)$$

The Hamilitonian is

$$H = \frac{1}{2}\tilde{p}T^{-1}p - L_0$$

Inverting T by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

with the algebra,

$$\frac{1}{ad-bc} = \frac{1}{m^2l^2(1+4ax^2) - m^2l^2(\cos\theta + 2ax\sin\theta)^2}$$

this is

$$= \frac{1}{m^2l^2(\sin^2\theta + 4ax^2 - 4ax\cos\theta\sin\theta - 4a^2x^2\sin^2\theta)}$$

$$= \frac{1}{m^2l^2(\sin^2\theta - 4ax\sin\theta\cos\theta + 4a^2x^2\cos^2\theta)}$$

which I'll introduce, for simplicity's sake, Y.

$$= \frac{1}{m^2l^2(\sin\theta - 2ax\cos\theta)^2} \equiv \frac{1}{m^2l^2Y}$$

So now we have

$$T^{-1} = \frac{1}{m^2l^2Y}\begin{pmatrix} ml^2 & -ml(\cos\theta + 2ax\sin\theta) \\ -ml(\cos\theta + 2ax\sin\theta) & m(1+4a^2x^2) \end{pmatrix}$$

$$T^{-1} = \frac{1}{mY}\begin{pmatrix} 1 & -(\cos\theta + 2ax\sin\theta)/l \\ -(\cos\theta + 2ax\sin\theta)/l & (1+4a^2x^2)/l^2 \end{pmatrix}$$

I want to introduce a new friend, lets call him J

$$J \equiv (\cos\theta + 2ax\sin\theta)$$

$$Y \equiv (\sin\theta - 2ax\cos\theta)^2$$

So,

$$T^{-1} = \frac{1}{mY}\begin{pmatrix} 1 & -J/l \\ -J/l & (1+4a^2x^2)/l^2 \end{pmatrix}$$

Proceed to derive the Hamiltonian,

$$H = \frac{1}{2}\tilde{p}T^{-1}p - L_0$$

we can go step by step,

$$T^{-1}p = \frac{1}{mY}\begin{pmatrix} 1 & -J/l \\ -J/l & (1+4a^2x^2/l^2 \end{pmatrix}\begin{pmatrix} p_x \\ p_\theta \end{pmatrix} = \frac{1}{mY}\begin{pmatrix} p_x - (J/l)p_\theta \\ (-J/l)p_x + (1+4a^2x^2/l^2)p_\theta \end{pmatrix}$$

and

$$\tilde{p}T^{-1}p = \frac{1}{mY}(p_x^2 - \frac{J}{l}p_\theta p_x - \frac{J}{l}p_\theta p_x + \frac{1+4a^2x^2}{l^2}p_\theta^2)$$

the full Hamiltonian is

$$H = \frac{1}{2mY}(p_x^2 - 2\frac{J}{l}p_\theta p_x + \frac{1+4a^2x^2}{l^2}p_\theta^2) + mg(ax^2 - l\cos\theta)$$

plugging in my Y and J

$$H = \frac{1}{2m(\sin\theta - 2ax\cos\theta)^2}(p_x^2 - 2\frac{\cos\theta + 2ax\sin\theta}{l}p_\theta p_x + \frac{1+4a^2x^2}{l^2}p_\theta^2) + mg(ax^2 - l\cos\theta)$$

Now to find the equations of motion. They are

$$\dot{x} = \frac{\partial H}{\partial p_x} \quad \dot{\theta} = \frac{\partial H}{\partial p_\theta} \quad \dot{p}_x = -\frac{\partial H}{\partial x} \quad \dot{p}_\theta = -\frac{\partial H}{\partial \theta}$$

The first two are easy, especially with my substitutions.

$$\dot{x} = \frac{1}{mY}[p_x - \frac{J}{l}p_\theta] = \frac{1}{m(\sin\theta - 2ax\cos\theta)^2}[p_x - \frac{\cos\theta + 2ax\sin\theta}{l}p_\theta]$$

$$\dot{\theta} = \frac{1}{mYl}[-Jp_x + \frac{1+4a^2x^2}{l}p_\theta] = \frac{1}{ml(\sin\theta - 2ax\cos\theta)^2}[-(\cos\theta + 2ax\sin\theta)p_x + \frac{1+4a^2x^2}{l}p_\theta]$$

But the next two are far more involved. I handled the partial with respect to x by taking the product rule between the two main pieces, the fraction out front, and mess inside the parenthesis that has p terms. I then broke each p term and began grouping them. Go slowly, and patiently. After taking the derivative before grouping, my $\dot{p}_x$ looked like this:

$$\dot{p}_x = -\frac{\partial H}{\partial x}$$

$$\frac{\partial H}{\partial x} = \frac{1}{2m(\sin\theta - 2ax\cos\theta)^2}[\frac{-4a\sin\theta}{l}p_\theta p_x + \frac{8a^2x}{l^2}p_\theta^2]$$

$$-\frac{-2(-2a\cos\theta)}{2m(\sin\theta - 2ax\cos\theta)^3}[p_x^2 - 2\frac{\cos\theta + 2ax\sin\theta}{l}p_\theta p_x + \frac{1+4a^2x^2}{l^2}p_\theta^2] + 2mgax$$

Now start simplifying. Lets group the p terms.

$$\frac{4a(\cos\theta + 2ax\sin\theta)}{2ml^2[\sin\theta - 2ax\cos\theta]^3}p_\theta^2$$

$$\frac{4a\cos\theta}{2m[\sin\theta - 2ax\cos\theta]^3}p_x^2$$

and the longest one..

$$\frac{2a}{lm[\sin\theta - 2ax\cos\theta]^3}[\sin^2\theta - 2 - 2ax\cos\theta\sin\theta]p_\theta p_x$$

Adding them all up yields, for $\dot{p}_x$:

$$-\frac{\partial H}{\partial x} = -\frac{2a}{m[\sin\theta - 2ax\cos\theta]^3}[\cos\theta p_x^2 + \frac{\cos\theta + 2ax\sin\theta}{l^2}p_\theta^2 - \frac{2 - \sin^2\theta + 2ax\sin\theta\cos\theta}{l}p_x p_\theta] - 2mgax$$

Now for the next one, $\dot{p}_\theta$:

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta}$$

Taking the derivative, you get a monster, of course

$$\frac{\partial H}{\partial \theta} = \frac{1}{2m[\sin\theta - 2ax\cos\theta]^2}[\frac{2\sin\theta}{l} - \frac{4ax\cos\theta}{l}]p_\theta p_x$$

$$+[p_x^2 - 2\frac{\cos\theta + 2ax\sin\theta}{l}p_\theta p_x + \frac{1 + 4a^2x^2}{l^2}p_\theta^2][\frac{-2(\cos\theta + 2ax\sin\theta)}{2m[\sin\theta - 2ax\cos\theta]^3} + mgl\sin\theta$$

separating terms..

$$\frac{-(\cos\theta + 2ax\sin\theta)}{m[\sin\theta - 2ax\cos\theta]^3}p_x^2$$

$$\frac{\cos\theta + 2ax\sin\theta}{m[\sin\theta - 2ax\cos\theta]^3}\frac{1 + 4a^2x^2}{l^2}p_\theta^2$$

and the longest one...

$$[\frac{(\sin\theta - 2ax\cos\theta)^2}{lm[\sin\theta - 2ax\cos\theta]^3} + \frac{2(\cos\theta + 2ax\sin\theta)^2}{lm[\sin\theta - 2ax\cos\theta]^3}]p_\theta p_x$$

add them all up for the fourth equation of motion, $\dot{p}_\theta$

$$-\frac{\partial H}{\partial \theta} = \frac{1}{m[\sin\theta - 2ax\cos\theta]^3}[(\cos\theta + 2ax\sin\theta)(p_x^2 + \frac{1 + 4a^2x^2}{l^2}p_\theta^2)$$

$$-\frac{[(\sin\theta - 2ax\cos\theta)^2 + 2(\cos\theta + 2ax\sin\theta)^2]}{l}p_\theta p_x$$

Together in all their glory:

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = \frac{1}{m[\sin\theta - 2ax\cos\theta]^3}[(\cos\theta + 2ax\sin\theta)(p_x^2 + \frac{1+4a^2x^2}{l^2}p_\theta^2)$$

$$-\frac{[(\sin\theta - 2ax\cos\theta)^2 + 2(\cos\theta + 2ax\sin\theta)^2]}{l}p_\theta p_x$$

$$\dot{p}_x = -\frac{\partial H}{\partial x} = -\frac{2a}{m[\sin\theta - 2ax\cos\theta]^3}[\cos\theta p_x^2 + \frac{\cos\theta + 2ax\sin\theta}{l^2}p_\theta^2 - \frac{2-\sin^2\theta + 2ax\sin\theta\cos\theta}{l}p_x p_\theta] - 2mgax$$

$$\dot{x} = \frac{1}{m(\sin\theta - 2ax\cos\theta)^2}[p_x - \frac{\cos\theta + 2ax\sin\theta}{l}p_\theta]$$

$$\dot{\theta} = \frac{1}{ml(\sin\theta - 2ax\cos\theta)^2}[-(\cos\theta + 2ax\sin\theta)p_x + \frac{1+4a^2x^2}{l}p_\theta]$$

8.24
A uniform cylinder of radius a and density ρ is mounted so as to rotate freely around a vertical axis. On the outside of the cylinder is a rigidly fixed uniform spiral or helical track along which a mass point m can slide without friction. Suppose a particle starts at rest at the top of the cylinder and slides down under the influence of gravity. Using any set of coordinates, arrive at a Hamiltonian for the combined system of particle and cylinder, and solve for the motion of the system.

Answer:

My generalized coordinates will be θ, the rotational angle of the particle with respect to the cylinder, and ϕ the rotational angle of the cylinder. The moment of inertia of a cylinder is

$$I = \frac{1}{2}Ma^2 = \frac{1}{2}\rho\pi h a^4$$

There are three forms of kinetic energy in the Lagrangian. The rotational energy of the cylinder, the rotational energy of the particle, and the translational kinetic energy of the particle. The only potential energy of the system will be the potential energy due to the height of the particle. The hardest part of this Lagrangian to understand is likely the translational energy due to the particle. The relationship between height and angle of rotational for a helix is

$$h = c\theta$$

Where c is the distance between the coils of the helix. MathWorld gives a treatment of this under helix. Understand that if the cylinder was not rotating then the rotational kinetic energy of the particle would merely be $\frac{m}{2}a^2\dot{\theta}^2$, but the rotation of the cylinder is adding an additional rotation to the particle's position. Lets write down the Lagrangian,

$$L = \frac{1}{2}I\dot{\phi}^2 + \frac{m}{2}[a^2(\dot{\theta}+\dot{\phi})^2 + c^2\dot{\theta}^2] + mgc\theta$$

This is

$$L = L_0 + \frac{1}{2}\tilde{\dot{q}}\, T\dot{q}$$

Solve for T.

$$T = \begin{pmatrix} ma^2 + mc^2 & ma^2 \\ ma^2 & I + ma^2 \end{pmatrix} \quad \dot{q} = \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Using the same 2 by 2 inverse matrix form from the previous problem, we may solve for T^{-1}.

$$T^{-1} = \frac{1}{(ma^2+mc^2)(I+ma^2) - m^2a^4} \begin{pmatrix} I + ma^2 & -ma^2 \\ -ma^2 & m(a^2+c^2) \end{pmatrix}$$

Now we can find the Hamiltonian.

$$H = \frac{1}{2}\tilde{p}\, T^{-1}p - L_0$$

This is

$$H = \frac{p_\theta^2(I+ma^2) - 2ma^2p_\theta p_\phi + p_\phi^2 m(a^2+c^2)}{2[m(a^2+c^2)(I+ma^2) - m^2a^4]} - mgc\theta$$

From the equations of motion, we can solve for the motion of the system. (duh!) Here are the EOM:

$$-\frac{\partial H}{\partial \theta} = \dot{p}_\theta = mgc$$

$$-\frac{\partial H}{\partial \phi} = \dot{p}_\phi = 0$$

$$\frac{\partial H}{\partial p_\theta} = \dot{\theta} = \frac{(I+ma^2)p_\theta - ma^2p_\phi}{m(a^2+c^2)(I+ma^2) - m^2a^4}$$

$$\frac{\partial H}{\partial p_\phi} = \dot{\phi} = \frac{-ma^2p_\theta + p_\phi m(a^2+c^2)}{m(a^2+c^2)(I+ma^2) - m^2a^4}$$

To solve for the motion, lets use the boundary conditions. $\dot{\theta}(0) = \dot{\phi}(0) = 0$ leads to $p_\phi(0) = p_\theta(0) = 0$ leads to

$$p_\theta = mgct \quad p_\phi = 0$$

Pluggin and chuggin into $\dot\theta$ and $\dot\phi$ and integrating, yields the motion

$$\phi = \frac{-m^2a^2gct^2}{2[m(a^2+c^2)(I+ma^2)-m^2a^4]}$$

$$\theta = \frac{(I+ma^2)mgct^2}{2[m(a^2+c^2)(I+ma^2)-m^2a^4]}$$

If we plug in $I = \frac{1}{2}Ma^2$ where M is the mass of the cylinder, we obtain

$$\phi = \frac{-mgct^2}{2[mc^2+\frac{1}{2}M(a^2+c^2)]}$$

$$\theta = \frac{(m+\frac{1}{2}M)gct^2}{2[mc^2+\frac{1}{2}M(a^2+c^2)]}$$

8.25
Suppose that in the previous exercise the cylinder is constrained to rotate uniformly with angular frequency ω. Set up the Hamiltonian for the particle in an inertial system of coordinates and also in a system fixed in the rotating cylinder. Identify the physical nature of the Hamiltonian in each case and indicate whether or not the Hamiltonians are conserved.

Answer:

In the laboratory system, the particle moves through an angle $\psi = \theta + \phi$. The cylinder moves uniformly, $\phi = \omega t$, so the kinetic energy

$$T = \frac{1}{2}ma^2(\dot\theta+\dot\phi)^2 + \frac{1}{2}mc^2\dot\theta^2$$

may be expressed

$$T = \frac{1}{2}ma^2\dot\psi^2 + \frac{1}{2}mc^2(\dot\psi-\omega)^2$$

The potential energy may be written

$$U = -mgc(\psi - \omega t)$$

So we have

$$L = \frac{1}{2}m(a^2\dot\psi^2 + c^2(\dot\psi-\omega)^2) + mgc(\psi-\omega t)$$

$$\frac{\partial L}{\partial \dot q} = p = ma^2\dot\psi + mc^2(\dot\psi - \omega)$$

and with

$$H = \frac{1}{2}(\tilde p - a)T^{-1}(\tilde p - a) - L_0$$

we find T^{-1} from

$$L = \frac{1}{2}\tilde{\dot{q}}\, T^{-1}\dot{q} + \dot{q}a + L_0$$

We can see things better if we spread out L

$$L = \frac{1}{2}ma^2\dot{\psi}^2 + \frac{1}{2}mc^2\dot{\psi}^2 - mc^2\omega\dot{\psi} + \frac{1}{2}mc^2\omega^2 + mgc(\psi - \omega t)$$

so

$$L_0 = \frac{1}{2}mc^2\omega^2 + mgc(\psi - \omega t)$$

and

$$T = [ma^2 + mc^2]$$

$$T^{-1} = \frac{1}{m(a^2 + c^2)}$$

Therefore, for our Hamiltonian, we have

$$H_{lab} = \frac{(p - mc^2\omega)^2}{2m(a^2 + c^2)} - \frac{mc^2\omega^2}{2} - mgc(\psi - \omega t)$$

This is dependent on time, therefore it is not the total energy.

For the Hamiltonian in the rotating cylinder's frame, we express the movement in terms of the angle θ this is with respect to the cylinder.

$$\psi = \theta + \phi = \theta + \omega t$$
$$\dot{\psi} = \dot{\theta} + \dot{\phi} = \dot{\theta} + \omega$$

$$T = \frac{1}{2}ma^2(\dot{\theta} + \omega)^2 + \frac{1}{2}mc^2\dot{\theta}^2$$
$$U = -mgc\theta$$

$$L = \frac{1}{2}ma^2(\dot{\theta} + \omega)^2 + \frac{1}{2}mc^2\dot{\theta}^2 + mgc\theta$$
$$L = \frac{1}{2}\tilde{\dot{q}}\, T\dot{q} + \dot{q}a + L_0$$

Spread out L

$$L = \frac{1}{2}[ma^2 + mc^2]\dot{\theta}^2 + ma^2\dot{\theta}\omega + \frac{1}{2}ma^2\omega^2 + mgc\theta$$

It becomes clear that

$$T = [ma^2 + mc^2]$$

$$T^{-1} = \frac{1}{ma^2 + mc^2}$$

$$L_0 = \frac{1}{2}ma^2\omega + mgc\theta$$

Using again,

$$H = \frac{1}{2}(p - a)T^{-1}(p - a) - L_0$$

we may write

$$H = \frac{(p - ma^2\omega)^2}{2m(a^2 + c^2)} - \frac{1}{2}ma^2\omega - mgc\theta$$

This is not explicitly dependent on time, it is time-independent, thus conserved.

Homework 10: # 9.2, 9.6, 9.16, 9.31

Michael Good

Nov 2, 2004

9.2
Show that the transformation for a system of one degree of freedom,

$$Q = q\cos\alpha - p\sin\alpha$$

$$P = q\sin\alpha + p\cos\alpha$$

satisfies the symplectic condition for any value of the parameter α. Find a generating function for the transformation. What is the physical significance of the transformation for $\alpha = 0$? For $\alpha = \pi/2$? Does your generating function work for both of these cases?

Answer:

The symplectic condition is met if

$$MJ\tilde{M} = J$$

We can find M from

$$\dot{\zeta} = M\dot{\eta}$$

which is

$$\begin{pmatrix} \dot{Q} \\ \dot{P} \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix}$$

We know J to be

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Solving $MJ\tilde{M}$ we get

$$M(J\tilde{M}) = M \begin{pmatrix} -\sin\alpha & \cos\alpha \\ -\cos\alpha & -\sin\alpha \end{pmatrix}$$

$$MJ\tilde{M} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} -\sin\alpha & \cos\alpha \\ -\cos\alpha & -\sin\alpha \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Therefore

$$M J \tilde{M} = J$$

and the symplectic condition is met for this transformation. To find the generating function, I will first attempt an F_1 type and proceed to solve, and check at the end if there are problems with it. Rearranging to solve for $p(Q, q)$ we have

$$p = -\frac{Q}{\sin\alpha} + \frac{q\cos\alpha}{\sin\alpha}$$

The related equation for F_1 is

$$p = \frac{\partial F_1}{\partial q}$$

Integrating for F_1 yields

$$F_1 = -\frac{Qq}{\sin\alpha} + \frac{q^2\cos\alpha}{2\sin\alpha} + g(Q)$$

Solve the other one, that is $P(Q, q)$, it along with its relevant equation is

$$P = q\sin\alpha - \frac{Q\cos\alpha}{\sin\alpha} + \frac{q\cos^2\alpha}{\sin\alpha}$$

$$P = -\frac{\partial F_1}{\partial Q}$$

Integrating

$$-F_1 = qQ\sin\alpha - \frac{Q^2}{2}\cot\alpha + qQ(\frac{1}{\sin\alpha} - \sin\alpha) + h(q)$$

$$-F_1 = -\frac{Q^2}{2}\cot\alpha + \frac{qQ}{\sin\alpha} + h(q)$$

$$F_1 = \frac{Q^2}{2}\cot\alpha - \frac{qQ}{\sin\alpha} + h(q)$$

Using both F_1's we find

$$F_1 = -\frac{Qq}{\sin\alpha} + \frac{1}{2}(q^2 + Q^2)\cot\alpha$$

This has a problem. It blows up, sky high, when $\alpha = n\pi$. But otherwise its ok, lets put the condition, $\alpha \neq n\pi$. If we solve for F_2 we may be able to find out what the generating function is, and have it work for the holes, $\alpha = n\pi$. $F_2(q, P, t)$'s relevant equations are

$$p = \frac{\partial F_2}{\partial q}$$

$$p = \frac{P}{\cos\alpha} - \frac{q\sin\alpha}{\cos\alpha}$$
$$F_2 = \frac{Pq}{\cos\alpha} - \frac{q^2}{2}\tan\alpha + f(P)$$

and

$$Q = \frac{\partial F_2}{\partial P}$$
$$Q = q\cos\alpha - (P - q\sin\alpha)\tan\alpha$$
$$F_2 = qP\cos\alpha - \frac{P^2}{2}\tan\alpha + qP\frac{\sin^2\alpha}{\cos\alpha} + g(q)$$
$$F_2 = qP(\cos\alpha + \frac{1}{\cos\alpha} - \cos\alpha) - \frac{P^2}{2}\tan\alpha + g(q)$$
$$F_2 = \frac{qP}{\cos\alpha} - \frac{P^2}{2}\tan\alpha + g(q)$$

So therefore

$$F_2 = -\frac{1}{2}(q^2 + P^2)\tan\alpha + \frac{qP}{\cos\alpha}$$

This works for $\alpha = n\pi$ but blows sky high for $\alpha = (n+\frac{1}{2})\pi$. So I'll put a condition on F_2 that $\alpha \neq (n+\frac{1}{2})\pi$. The physical significance of this transformation for $\alpha = 0$ is easy to see cause we get

$$Q = q\cos 0 - p\sin 0 = q$$
$$P = q\sin 0 - p\cos 0 = p$$

This is just the identity transformation, or no rotation. For $\alpha = \pi/2$ we get

$$Q = q\cos\frac{\pi}{2} - p\sin\frac{\pi}{2} = -p$$
$$P = q\sin\frac{\pi}{2} - p\cos\frac{\pi}{2} = q$$

Where the p's and q's have been exchanged.

9.6 The transformation equations between two sets of coordinates are

$$Q = log(1 + q^{1/2} \cos p)$$
$$P = 2(1 + q^{1/2} \cos p) q^{1/2} \sin p$$

- Show directly from these transformation equations that Q, P are canonical variables if q and p are.
- Show that the function that generates this transformation is

$$F_3 = -(e^Q - 1)^2 \tan p$$

Answer:

Q and P are considered canonical variables if these transformation equations satisfy the symplectic condition.

$$MJ\tilde{M} = J$$

Finding M:

$$\dot{\zeta} = M\dot{\eta}$$

$$\begin{pmatrix} \dot{Q} \\ \dot{P} \end{pmatrix} = M \begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix}$$

$$M_{ij} = \frac{\partial \zeta_i}{\partial \eta_j} \qquad M = \begin{pmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{pmatrix}$$

$$\frac{\partial Q}{\partial q} = \frac{q^{-1/2} \cos p}{2(1 + q^{1/2} \cos p)}$$

$$\frac{\partial Q}{\partial p} = \frac{-q^{1/2} \sin p}{1 + q^{1/2} \cos p}$$

$$\frac{\partial P}{\partial q} = q^{-1/2} \sin p + 2 \cos p \sin p$$

$$\frac{\partial P}{\partial p} = 2q^{1/2} \cos p + 2q \cos^2 p - 2q \sin^2 p$$

Remembering

$$\cos^2 A - \sin^2 A = \cos 2A \quad and \quad 2 \sin A \cos A = \sin 2A$$
$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$

we can proceed with ease.

$$JM = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{q^{-1/2}\cos p}{2(1+q^{1/2}\cos p)} & \frac{-q^{1/2}\sin p}{1+q^{1/2}\cos p} \\ q^{-1/2}\sin p + \sin 2p & 2q^{1/2}\cos p + 2q\cos 2p \end{pmatrix}$$

$$JM = \begin{pmatrix} q^{-1/2}\sin p + \sin 2p & 2q^{1/2}\cos p + 2q\cos 2p \\ -\frac{q^{-1/2}\cos p}{2(1+q^{1/2}\cos p)} & \frac{q^{1/2}\sin p}{1+q^{1/2}\cos p} \end{pmatrix}$$

Now

$$\tilde{M}JM = \begin{pmatrix} \frac{q^{-1/2}\cos p}{2(1+q^{1/2}\cos p)} & q^{-1/2}\sin p + \sin 2p \\ \frac{-q^{1/2}\sin p}{1+q^{1/2}\cos p} & 2q^{1/2}\cos p + 2q\cos 2p \end{pmatrix} \begin{pmatrix} q^{-1/2}\sin p + \sin 2p & 2q^{1/2}\cos p + 2q\cos 2p \\ -\frac{q^{-1/2}\cos p}{2(1+q^{1/2}\cos p)} & \frac{q^{1/2}\sin p}{1+q^{1/2}\cos p} \end{pmatrix}$$

You may see that the diagonal terms disappear, and we are left with some algebra for the off-diagonal terms.

$$\tilde{M}JM = \begin{pmatrix} 0 & messy \\ ugly & 0 \end{pmatrix}$$

Lets solve for *ugly*.

$$ugly = \frac{-q^{1/2}\sin p}{1+q^{1/2}\cos p}(q^{-1/2}\sin p + \sin 2p) - \frac{q^{-1/2}\cos p}{2(1+q^{1/2}\cos p)}(2q^{1/2}\cos p + 2q\cos 2p)$$

$$ugly = \frac{-\sin^2 p - q^{1/2}\sin p \sin 2p - \cos^2 p - q^{1/2}\cos p\cos 2p}{1+q^{1/2}\cos p}$$

$$ugly = \frac{-(1+q^{1/2}(\cos 2p\cos p + \sin 2p \sin p))}{1+q^{1/2}\cos p}$$

$$ugly = \frac{-(1+q^{1/2}\cos p)}{1+q^{1/2}\cos p} = -1$$

Not so ugly anymore, eh? Suddenly ugly became pretty. The same works for *messy* except it becomes positive 1 because it has no negative terms out front. So finally we get

$$\tilde{M}JM = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = J$$

which is the symplectic condition, which proves Q and P are canonical variables. To show that

$$F_3 = -(e^Q - 1)^2 \tan p$$

generates this transformation we may take the relevant equations for F_3, solve them, and then solve for our transformation equations.

$$q = -\frac{\partial F_3}{\partial p} = -[-(e^Q - 1)^2 \sec^2 p]$$

$$P = -\frac{\partial F_3}{\partial Q} = -[-2(e^Q - 1)\tan p]e^Q$$

Solving for Q

$$q = (e^Q - 1)^2 \sec^2 p$$

$$1 + \frac{\sqrt{q}}{\sqrt{\sec^2 p}} = e^Q$$

$$Q = \ln(1 + q^{1/2}\cos p)$$

This is one of our transformation equations, now lets plug this into the expression for P and put P in terms of q and p to get the other one.

$$P = 2(1 + q^{1/2}\cos p - 1)\tan p(1 + q^{1/2}\cos p)$$

$$P = 2q^{1/2}\sin p(1 + q^{1/2}\cos p)$$

Thus F_3 is the generating function of our transformation equations.

9.16
For a symmetric rigid body, obtain formulas for evaluating the Poisson brackets

$$[\dot{\phi}, f(\theta, \phi, \psi)] \qquad [\dot{\psi}, f(\theta, \phi, \psi)]$$

where θ, ϕ, and ψ are the Euler angles, and f is any arbitrary function of the Euler angles.

Answer:

Poisson brackets are defined by

$$[u, v]_{q,p} = \frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial v}{\partial q_i}$$

From Goldstein's section on Euler angles, we learned

$$\dot{\phi} = \frac{I_1 b - I_1 a\cos\theta}{I_1 \sin^2\theta} = \frac{p_\phi - p_\psi \cos\theta}{I_1 \sin^2\theta}$$

So calculating

$$[\dot{\phi}, f] = [\frac{p_\phi - p_\psi\cos\theta}{I_1\sin^2\theta}, f]$$

Note that $f = f(\theta, \phi, \psi)$ and not of momenta. So our definition becomes

$$[\dot{\phi}, f] = -\frac{\partial \dot{\phi}}{\partial p_i}\frac{\partial f}{\partial q_i}$$

Taking only two derivatives because $\dot{\phi}$ doesn't depend on p_θ. We get

$$[\dot{\phi}, f] = (-\frac{1}{I_1 \sin^2\theta}\frac{\partial f}{\partial \phi}) + (\frac{\cos\theta}{I_1 \sin^2\theta}\frac{\partial f}{\partial \psi})$$

$$[\dot{\phi}, f] = \frac{1}{I_1 \sin^2\theta}(-\frac{\partial f}{\partial \psi} + \frac{\partial f}{\partial \psi}\cos\theta)$$

For the next relation,

$$[\dot{\psi}, f] = -\frac{\partial \dot{\psi}}{\partial p_i}\frac{\partial f}{\partial q_i}$$

and

$$\dot{\psi} = \frac{p_\psi}{I_3} - \frac{p_\phi - p_\psi \cos\theta}{I_1 \sin^2\theta}\cos\theta$$

This yields

$$[\dot{\psi}, f] = -(\frac{1}{I_3} + \frac{\cos^2\theta}{I_1 \sin^2\theta})\frac{\partial f}{\partial \psi} + -(-\frac{\cos\theta}{I_1 \sin^2\theta}\frac{\partial f}{\partial \psi})$$

$$[\dot{\psi}, f] = -(\frac{I_1 \sin^2\theta}{I_3 I_1 \sin^2\theta} + \frac{I_3 \cos^2\theta}{I_3 I_1 \sin^2\theta})\frac{\partial f}{\partial \psi} + \frac{I_3 \cos\theta}{I_3 I_1 \sin^2\theta}\frac{\partial f}{\partial \phi}$$

$$[\dot{\psi}, f] = \frac{1}{I_3 I_1 \sin^2\theta}(I_3 \cos\theta \frac{\partial f}{\partial \phi} - (I_3 \cos^2\theta + I_1 \sin^2\theta)\frac{\partial f}{\partial \psi})$$

Both together, in final form

$$[\dot{\phi}, f] = \frac{1}{I_1 \sin^2\theta}(-\frac{\partial f}{\partial \psi} + \frac{\partial f}{\partial \psi}\cos\theta)$$

$$[\dot{\psi}, f] = \frac{1}{I_3 I_1 \sin^2\theta}(I_3 \cos\theta \frac{\partial f}{\partial \phi} - (I_3 \cos^2\theta + I_1 \sin^2\theta)\frac{\partial f}{\partial \psi})$$

9.31
Show by the use of Poisson brackets that for one-dimensional harmonic oscillator there is a constant of the motion u defined as

$$u(q, p, t) = \ln(p + im\omega q) - i\omega t, \omega = \sqrt{\frac{k}{m}}$$

What is the physical significance of this constant of motion?

Answer:

We have

$$\frac{du}{dt} = [u, H] + \frac{\partial u}{\partial t}$$

which we must prove equals zero if u is to be a constant of the motion. The Hamiltonian is

$$H(q, p) = \frac{p^2}{2m} + \frac{kq^2}{2}$$

So we have

$$\frac{du}{dt} = \frac{\partial u}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial u}{\partial p}\frac{\partial H}{\partial q} + \frac{\partial u}{\partial t}$$

$$\frac{du}{dt} = \frac{im\omega}{p + im\omega q}(\frac{p}{m}) - \frac{1}{p + im\omega q}(kq) - i\omega$$

$$\frac{du}{dt} = \frac{i\omega p - kq}{p + im\omega q} - i\omega = \frac{i\omega p - m\omega^2 q}{p + im\omega q} - i\omega$$

$$\frac{du}{dt} = i\omega\frac{p + i\omega mq}{p + im\omega q} - i\omega = i\omega - i\omega$$

$$\frac{du}{dt} = 0$$

Its physical significance relates to phase.

Show Jacobi's Identity holds. Show

$$[f, gh] = g[f, h] + [f, g]h$$

where the brackets are Poisson.

Answer:

Goldstein verifies Jacobi's identity

$$[u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0$$

using an efficient notation. I will follow his lead. If we say

$$u_i \equiv \frac{\partial u}{\partial \eta_i} \qquad v_{ij} \equiv \frac{\partial v}{\partial \eta_i \partial \eta_j}$$

Then a simple way of expressing the Poisson bracket becomes apparent

$$[u, v] = u_i J_{ij} v_j$$

This notation becomes valuable when expressing the the double Poisson bracket. Here we have

$$[u,[v,w]] = u_i J_{ij}[v,w]_j = u_i J_{ij}(v_k J_{kl} w_l)_j$$

Taking the partial with respect to η_j we use the product rule, remembering J_{kl} are just constants,

$$[u,[v,w]] = u_i J_{ij}(v_{kj} J_{kl} w_l + v_k J_{kl} w_{lj})$$

doing this for the other two double Poisson brackets, we get 4 more terms, for a total of 6. Looking at one double partial term, w we see there are only two terms that show up

$$J_{ij}J_{kl}u_i v_k w_{lj} \quad and \quad J_{ji}J_{kl}u_i v_k w_{jl}$$

The first from $[u,[v,w]]$ and the second from $[v,[w,u]]$. Add them up, realizing order of partial is immaterial, and J is antisymmetric:

$$(J_{ij} + J_{ji})J_{kl}u_i v_k w_{lj} = 0$$

All the other terms are made of second partials of u or v and disappear in the same manner. Therefore

$$[u,[v,w]] + [v,[w,u]] + [w,[u,v]] = 0$$

Its ok to do the second property the long way:

$$[f,gh] = \frac{\partial f}{\partial q_i}\frac{\partial (gh)}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial (gh)}{\partial q_i}$$

$$[f,gh] = \frac{\partial f}{\partial q_i}(\frac{\partial g}{\partial p_i}h + g\frac{\partial h}{\partial p_i}) - \frac{\partial f}{\partial p_i}(g\frac{\partial h}{\partial q_i} + \frac{\partial g}{\partial q_i}h)$$

Grouping terms

$$[f,gh] = \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}h - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}h + g\frac{\partial f}{\partial q_i}\frac{\partial h}{\partial p_i} - g\frac{\partial f}{\partial p_i}\frac{\partial h}{\partial q_i}$$

$$[f,gh] = [f,g]h + g[f,h]$$

Homework 11: # 10.7 b, 10.17, 10.26

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10.7

- A single particle moves in space under a conservative potential. Set up the Hamilton-Jacobi equation in ellipsoidal coordinates u, v, ϕ defined in terms of the usual cylindrical coordinates r, z, ϕ by the equations.

$$r = a \sinh v \sin u \qquad z = a \cosh v \cos u$$

  For what forms of $V(u, v, \phi)$ is the equation separable.

- Use the results above to reduce to quadratures the problem of point particle of mass m moving in the gravitational field of two unequal mass points fixed on the z axis a distance $2a$ apart.

Answer:

Let's obtain the Hamilton Jacobi equation. This will be used to reduce the problem to quadratures. This is an old usage of the word quadratures, and means to just get the problem into a form where the only thing left to do is take an integral.

Here

$$T = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}m\dot{z}^2 + \frac{1}{2}mr^2\dot{\phi}^2$$

$$r = a \sinh v \sin u$$

$$\dot{r} = a \cosh v \sin u \dot{v} + a \sinh v \cos u \dot{u}$$

$$z = a \cosh v \cos u$$

$$\dot{z} = a \sinh v \cos u \dot{v} - a \cosh v \sin u \dot{u}$$

Here

$$\dot{r}^2 + \dot{z}^2 = a^2(\cosh^2 v \sin^2 u + \sinh^2 v \cos^2 u)(\dot{v}^2 + \dot{u}^2) = a^2(\sin^2 u + \sinh^2 v)(\dot{v}^2 + \dot{u}^2)$$

To express in terms of momenta use

$$p_v = \frac{\partial L}{\partial \dot{v}} = ma^2(\sin^2 u + \sinh^2 v)\dot{v}$$

$$p_u = \frac{\partial L}{\partial \dot{u}} = ma^2(\sin^2 u + \sinh^2 v)\dot{u}$$

because the potential does not depend on $\dot{v}$ or $\dot{u}$. The cyclic coordinate ϕ yields a constant I'll call α_ϕ

$$p_\phi = mr^2\dot{\phi} = \alpha_\phi$$

So our Hamiltonian is

$$H = \frac{p_v^2 + p_u^2}{2ma^2(\sin^2 u + \sinh^2 v)} + \frac{p_\phi^2}{2ma^2 \sinh^2 v \sin^2 u} + V$$

To find our Hamilton Jacobi expression, the principle function applies

$$S = W_u + W_v + \alpha_\phi \phi - Et$$

So our Hamilton Jacobi equation is

$$\frac{1}{2ma^2(\sin^2 u + \sinh^2 v)}[(\frac{\partial W_u}{\partial u})^2 + (\frac{\partial W_v}{\partial v})^2] + \frac{1}{2ma^2 \sinh^2 v \sin^2 u}(\frac{\partial W_\phi}{\partial \phi})^2 + V(u, v, \phi) = E$$

This is

$$\frac{1}{2ma^2}[(\frac{\partial W_u}{\partial u})^2 + (\frac{\partial W_v}{\partial v})^2] + \frac{1}{2ma^2}(\frac{1}{\sinh^2 v} + \frac{1}{\sin^2 u})\alpha_\phi^2 + (\sin^2 u + \sinh^2 v)V(u, v, \phi) = (\sin^2 u + \sinh^2 v)E$$

A little bit more work is necessary. Once we solve for $V(u, v, \phi)$ we can then separate this equation into u, v and ϕ parts, at which point we will have only integrals to take.

I suggest drawing a picture, with two point masses on the z axis, with the origin being between them, so they are each a distance a from the origin. The potential is then formed from two pieces

$$V = -\frac{GmM_1}{|\vec{r} - a\hat{z}|} - \frac{GmM_2}{|\vec{r} + a\hat{z}|}$$

To solve for the denominators use the Pythagorean theorem, remembering we are in cylindrical coordinates,

$$|\vec{r} \mp a\hat{z}|^2 = (z \mp a)^2 + r^2$$

Using the results from part (a) for r and z,

$$|\vec{r} \mp a\hat{z}|^2 = a^2(\cosh v \cos u \mp 1)^2 + a^2 \sinh^2 v \sin^2 u$$

$$|\vec{r} \mp a\hat{z}|^2 = a^2(\cosh^2 v \cos^2 u \mp 2\cosh v \cos u + 1 + \sinh^2 v \sin^2 u)$$

Lets rearrange this to make it easy to see the next step,

$$|\vec{r} \mp a\hat{z}|^2 = a^2(\sinh^2 v \sin^2 u + \cosh^2 v \cos^2 u + 1 \mp 2\cosh v \cos u)$$

Now convert the $\sin^2 u = 1 - \cos^2 u$ and convert the $\cosh^2 v = 1 + \sinh^2 v$

$$|\vec{r} \mp a\hat{z}|^2 = a^2(\sinh^2 v + \cos^2 u + 1 \mp 2\cosh v \cos u)$$

Add the 1 and $\cosh^2 v$

$$|\vec{r} \mp a\hat{z}|^2 = a^2(\cosh^2 v + \cos^2 u \mp 2\cosh v \cos u)$$
$$|\vec{r} \mp a\hat{z}|^2 = (a(\cosh v \mp \cos u))^2$$

So our potential is now

$$V = -\frac{GmM_1}{a(\cosh v - \cos u)} - \frac{GmM_2}{a(\cosh v + \cos u)}$$

$$V = -\frac{1}{a}\frac{GmM_1(\cosh v + \cos u) + GmM_2(\cosh v - \cos u)}{\cosh^2 v - \cos^2 u}$$

Note the very helpful substitution

$$\cosh^2 v - \cos^2 u = \sin^2 u + \sinh^2 v$$

Allowing us to write V

$$V = -\frac{1}{a}\frac{GmM_1(\cosh v + \cos u) + GmM_2(\cosh v - \cos u)}{\sin^2 u + \sinh^2 v}$$

Plug this into our Hamilton Jacobi equation, and go ahead and separate out u and v terms, introducing another constant, A:

$$\frac{1}{2ma^2}(\frac{\partial W_u}{\partial u})^2 + \frac{1}{2ma^2}\frac{\alpha_\phi^2}{\sin^2 u} - \frac{1}{a}Gm(M_1 - M_2)\cos u - E\sin^2 u = A$$

$$\frac{1}{2ma^2}(\frac{\partial W_v}{\partial v})^2 + \frac{1}{2ma^2}\frac{\alpha_\phi^2}{\sinh^2 v} - \frac{1}{a}Gm(M_1 - M_2)\cosh v - E\sinh^2 v = -A$$

The problem has been reduced to quadratures.

10.17
Solve the problem of the motion of a point projectile in a vertical plane, using the Hamilton-Jacobi method. Find both the equation of the trajectory and the dependence of the coordinates on time, assuming the projectile is fired off at time $t = 0$ from the origin with the velocity v_0, making an angle θ with the horizontal.

Answer:

I'm going to assume the angle is θ because there are too many α's in the problem to begin with. First we find the Hamiltonian,

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + mgy$$

Following the examples in section 10.2, we set up the Hamiltonian-Jacobi equation by setting $p = \partial S/\partial q$ and we get

$$\frac{1}{2m}(\frac{\partial S}{\partial x})^2 + \frac{1}{2m}(\frac{\partial S}{\partial y})^2 + mgy + \frac{\partial S}{\partial t} = 0$$

The principle function is

$$S(x, \alpha_x, y, \alpha, t) = W_x(x, \alpha_x) + W_y(y, \alpha) - \alpha t$$

Because x is not in the Hamiltonian, it is cyclic, and a cyclic coordinate has the characteristic component $W_{q_i} = q_i\alpha_i$.

$$S(x, \alpha_x, y, \alpha, t) = x\alpha_x + W_y(y, \alpha) - \alpha t$$

Expressed in terms of the characteristic function, we get for our Hamiltonian-Jacobi equation

$$\frac{\alpha_x^2}{2m} + \frac{1}{2m}(\frac{\partial W_y}{\partial y})^2 + mgy = \alpha$$

This is

$$\frac{\partial W_y}{\partial y} = \sqrt{2m\alpha - \alpha_x^2 - 2m^2gy}$$

Integrated, we have

$$W_y(y, \alpha) = -\frac{1}{3m^2g}(2m\alpha - \alpha_x^2 - 2m^2gy)^{3/2}$$

Thus our principle function is

$$S(x, \alpha_x, y, \alpha, t) = x\alpha_x + -\frac{1}{3m^2g}(2m\alpha - \alpha_x^2 - 2m^2gy)^{3/2} - \alpha t$$

Solving for the coordinates,

$$\beta = \frac{\partial S}{\partial \alpha} = -\frac{1}{mg}(2m\alpha - \alpha_x^2 - 2m^2 gy)^{1/2} - t$$

$$\beta_x = \frac{\partial S}{\partial \alpha_x} = x + \frac{\alpha_x}{m^2 g}(2m\alpha - \alpha_x^2 - 2m^2 gy)^{1/2}$$

Solving for both $x(t)$ and $y(t)$ in terms of the constants β, β_x, α and α_x

$$y(t) = -\frac{g}{2}(t+\beta)^2 + \frac{\alpha}{mg} - \frac{\alpha_x^2}{2m^2 g}$$

$$x(t) = \beta_x + \frac{\alpha_x}{m}(-\frac{1}{mg}(2m\alpha - \alpha_x^2 - 2m^2 gy)^{1/2})$$

Our $x(t)$ is

$$x(t) = \beta_x + \frac{\alpha_x}{m}(\beta + t)$$

We can solve for our constants in terms of our initial velocity, and angle θ through initial conditions,

$$x(0) = 0 \rightarrow \beta_x = -\frac{\alpha_x}{m}\beta$$

$$y(0) = 0 \rightarrow -\frac{g}{2}\beta^2 + \frac{\alpha}{mg} - \frac{\alpha_x^2}{2m^2 g} = 0$$

$$\dot{x}(0) = v_0 \cos\theta = \frac{\alpha_x}{m}$$

$$\dot{y}(0) = v_0 \sin\theta = -g\beta$$

Thus we have for our constants

$$\beta = \frac{v_0 \sin\theta}{-g}$$

$$\beta_x = \frac{v_0^2}{g}\cos\theta \sin\theta$$

$$\alpha = \frac{mg}{2g}(v_0^2 \sin^2\theta + v_0^2 \cos^2\theta) = \frac{mv_0^2}{2}$$

$$\alpha_x = mv_0 \cos\theta$$

Now our $y(t)$ is

$$y(t) = -\frac{g}{2}(t + \frac{v_0 \sin\theta}{g})^2 + \frac{v_0^2}{g} - \frac{v_0^2 \cos^2\theta}{2g}$$

$$y(t) = -\frac{g}{2}t^2 + v_0 \sin\theta t - \frac{g}{2}\frac{v_0^2 \sin^2\theta}{g^2} + \frac{v_0^2}{g} - \frac{v_0^2 \cos^2\theta}{2g}$$

$$y(t) = -\frac{g}{2}t^2 + v_0 \sin\theta t$$

and for $x(t)$

$$x(t) = \frac{v_0^2}{g}\cos\theta\sin\theta + v_0\cos\theta\frac{v_0\sin\theta}{-g} + v_0\cos\theta t$$

$$x(t) = v_0\cos\theta t$$

Together we have

$$y(t) = -\frac{g}{2}t^2 + v_0 \sin\theta t$$

$$x(t) = v_0\cos\theta t$$

10.26
Set up the problem of the heavy symmetrical top, with one point fixed, in the Hamilton-Jacobi mehtod, and obtain the formal solution to the motion as given by Eq. (5.63).

Answer:

This is the form we are looking for.

$$t = \int_{u(0)}^{u(t)} \frac{du}{\sqrt{(1-u^2)(\alpha-\beta u)-(b-au)^2}}$$

Expressing the Hamiltonian in terms of momenta like we did in the previous problem, we get

$$H = \frac{p_\psi^2}{2I_3} + \frac{p_\theta^2}{2I_1} + \frac{(p_\phi - p_\psi\cos\theta)^2}{2I_1\sin^2\theta} + Mgh\cos\theta$$

Setting up the principle function, noting the cyclic coordinates, we see

$$S(\theta, E, \psi, \alpha_\psi, \phi, \alpha_\phi, t) = W_\theta(\theta, E) + \psi\alpha_\psi + \phi\alpha_\phi - Et$$

Using

$$\frac{\partial S}{\partial q} = p$$

we have for our Hamilton-Jacobi equation, solved for the partial S's

$$\frac{\alpha_\psi^2}{2I_3} + \frac{1}{2I_1}(\frac{\partial W_\theta}{\partial\theta})^2 + \frac{(\alpha_\phi - \alpha_\psi\cos\theta)^2}{2I_1\sin^2\theta} + Mgh\cos\theta = E$$

Turning this inside out:

$$\frac{\partial}{\partial\theta}W_\theta(\theta,E)=\sqrt{2I_1E-\frac{\alpha_\psi^2 I_1}{I_3}-\frac{(\alpha_\phi-\alpha_\psi\cos\theta)^2}{\sin^2\theta}-2I_1Mgh\cos\theta}$$

When integrated,

$$W_\theta=\int(2I_1E-\frac{\alpha_\psi^2 I_1}{I_3}-\frac{(\alpha_\phi-\alpha_\psi\cos\theta)^2}{\sin^2\theta}-2I_1Mgh\cos\theta)^{1/2}d\theta$$

Now we are in a position to solve

$$\beta_\theta=\frac{\partial S}{\partial E}=\frac{\partial W_\theta}{\partial E}-t$$

$$\frac{\partial W_\theta}{\partial E}=\beta_\theta+t=\int\frac{2I_1d\theta}{2(2I_1E-\frac{\alpha_\psi^2 I_1}{I_3}-\frac{(\alpha_\phi-\alpha_\psi\cos\theta)^2}{\sin^2\theta}-2I_1Mgh\cos\theta)^{1/2}}$$

Using the same constants Goldstein uses

$$\alpha=\frac{2E-\frac{\alpha_\psi^2}{I_3}}{I_1}=\frac{2E}{I_1}-\frac{\alpha_\psi^2}{I_3I_1}$$
$$\beta=\frac{2Mgl}{I_1}$$

where

$$\alpha_\phi=I_1b$$
$$\alpha_\psi=I_1a$$

and making these substitutions

$$\beta_\theta+t=\int\frac{I_1d\theta}{(I_1(2E-\frac{\alpha_\psi^2}{I_3})-I_1^2\frac{(b-a\cos\theta)^2}{\sin^2\theta}-I_12Mgh\cos\theta)^{1/2}}$$
$$\beta_\theta+t=\int\frac{d\theta}{(\alpha-\frac{(b-a\cos\theta)^2}{\sin^2\theta}-\beta\cos\theta)^{1/2}}$$

For time t, the value of θ is $\theta(t)$

$$t=\int_{\theta(0)}^{\theta(t)}\frac{d\theta}{(\alpha-\frac{(b-a\cos\theta)^2}{\sin^2\theta}-\beta\cos\theta)^{1/2}}$$

The integrand is the exact expression as Goldstein's (5.62). Making the substitution $u=\cos\theta$ we arrive home

$$t=\int_{u(0)}^{u(t)}\frac{du}{\sqrt{(1-u^2)(\alpha-\beta u)-(b-au)^2}}$$

Homework 12: # 10.13, 10.27, Cylinder

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10.13
A particle moves in periodic motion in one dimension under the influence of a potential $V(x) = F|x|$, where F is a constant. Using action-angle variables, find the period of the motion as a function of the particle's energy.

Solution:

Define the Hamiltonian of the particle

$$H \equiv E = \frac{p^2}{2m} + F|q|$$

Using the action variable definition, which is Goldstein's (10.82):

$$J = \oint p\, dq$$

we have

$$J = \oint \sqrt{2m(E - Fq)}\, dq$$

For F is greater than zero, we have only the first quadrant, integrated from $q = 0$ to $q = E/F$ (where $p = 0$). Multiply this by 4 for all of phase space and our action variable J becomes

$$J = 4\int_0^{E/F} \sqrt{2m}\sqrt{E - Fq}\, dq$$

A lovely u-substitution helps out nicely here.

$$u = E - Fq \qquad \rightarrow \qquad du = -F\, dq$$

$$J = 4\int_E^0 \sqrt{2m}u^{1/2}\frac{1}{-F}\, du$$

$$J = \frac{4\sqrt{2m}}{F}\int_0^E u^{1/2}\, du = \frac{8\sqrt{2m}}{3F}E^{3/2}$$

Goldstein's (10.95) may help us remember that

$$\frac{\partial H}{\partial J} = \nu$$

and because $E = H$ and $\tau = 1/\nu$,

$$\tau = \frac{\partial J}{\partial E}$$

This is

$$\tau = \frac{\partial}{\partial E}[\frac{8\sqrt{2m}}{3F}E^{3/2}]$$

And our period is

$$\tau = \frac{4\sqrt{2mE}}{F}$$

10.27
Describe the phenomenon of small radial oscillations about steady circular motion in a central force potential as a one-dimensional problem in the action-angle formalism. With a suitable Taylor series expansion of the potential, find the period of the small oscillations. Express the motion in terms of J and its conjugate angle variable.

Solution:

As a reminder, Taylor series go like

$$f(x) = f(a) + (x-a)f'(a) + \frac{1}{2!}(x-a)^2 f''(a) + ...$$

Lets expand around some r_0 for our potential,

$$U(r) = U(r_0) + (r-r_0)U'(r_0) + \frac{1}{2}(r-r_0)^2 U''(r_0) + ...$$

Using the form of the Hamiltonian, involving two degrees of freedom, in polar coordinates,(eq'n 10.65) we have

$$H = \frac{1}{2m}(p_r^2 + \frac{l^2}{r^2}) + V(r)$$

Defining a new equivalent potential, $U(r)$ the Hamiltonian becomes

$$H = \frac{1}{2m}p_r^2 + U(r)$$

The r_0 from above will be some minimum of $U(r)$,

$$U'(r_0) = -\frac{l^2}{mr_0^3} + V'(r_0) = 0$$

The second derivative is the only contribution

$$U'' = \frac{3l^2}{mr_0^4} + V''(r_0) = k$$

where $k > 0$ because we are at a minimum that is concave up. If there is a small oscillation about circular motion we may let

$$r = r_0 + \lambda$$

where λ will be very small compared to r_0. Thus our Hamiltonian becomes

$$H = \frac{1}{2m}p_r^2 + U(r_0 + \lambda)$$

This is

$$H = \frac{1}{2m}p_r^2 + U(r_0) + \frac{1}{2}(r - r_0)^2 U''(r_0)$$

$$H = \frac{1}{2m}p_r^2 + U(r_0) + \frac{1}{2}\lambda^2 U''(r_0) = E$$

If we use the small energy ϵ defined as

$$\epsilon = E - U(r_0)$$

We see

$$\epsilon = \frac{1}{2m}p_r^2 + \frac{1}{2}\lambda^2 k$$

This energy is the effect on the frequency, so following section 10.6

$$\epsilon = \frac{J\omega}{2\pi}$$

We have for the action variable

$$J = 2\pi\epsilon\sqrt{\frac{m}{k}}$$

and a period

$$\tau = \frac{\partial J}{\partial \epsilon} = 2\pi\sqrt{\frac{m}{k}}$$

with motion given by

$$r = r_0 + \sqrt{\frac{J}{\pi m\omega}}\sin 2\pi\omega$$

$$p_r = \sqrt{\frac{mJ\omega}{\pi}}\cos 2\pi\omega$$

A particle is constrained to the edge of a cylinder. It is released and bounces around the perimeter. Find the two frequencies of its motion using the action angle variable formulation.

Solution:

Trivially, we know the frequency around the cylinder to be its angular speed divided by 2π because it goes 2π radians in one revolution.

$$\nu_\theta = \frac{\dot{\theta}}{2\pi}$$

And also simply, we may find the frequency of its up and down bouncing through Newtonian's equation of motion.

$$h = \frac{1}{2}gt^2$$

$$t = \sqrt{\frac{2h}{g}}$$

Multiply this by 2 because the period will be measured from a point on the bottom of the cylinder to when it next hits the bottom of the cylinder again. The time it takes to fall is the same time it takes to bounce up, by symmetry.

$$T = 2\sqrt{\frac{2h}{g}} \quad \rightarrow \quad \nu_z = \frac{1}{2}\sqrt{\frac{g}{2h}}$$

To derive these frequencies via the action-angle formulation we first start by writing down the Hamiltonian for the particle.

$$H \equiv E = mgz + \frac{p_z^2}{2m} + \frac{p_\theta^2}{2mR^2}$$

Noting that p_θ is constant because there is no external forces on the system, and because θ does not appear in the Hamiltonian, therefore it is cyclic and its conjugate momentum is constant.

$$p_\theta = m\dot{\theta}R^2$$

we may write

$$J_\theta = 2\pi p_\theta$$

based on Goldstein's (10.101), and his very fine explanation. Breaking the energy into two parts, one for θ and one for z, we may express the E_θ part as a function of J_θ.

$$E_\theta = \frac{p_\theta^2}{2mR^2} = \frac{J_\theta^2}{4\pi^2 2mR^2}$$

The frequency is

$$\nu_\theta = \frac{\partial E_\theta}{\partial J_\theta} = \frac{J_\theta}{4\pi^2 mR^2}$$

$$\nu_\theta = \frac{J_\theta}{4\pi^2 mR^2} = \frac{2\pi p_\theta}{4\pi^2 mR^2} = \frac{p_\theta}{2\pi mR^2} = \frac{m\dot{\theta}R^2}{2\pi mR^2}$$

Thus we have

$$\nu_\theta = \frac{\dot{\theta}}{2\pi}$$

The second part is a bit more involved algebraically. Expressing the energy for z:

$$E_z = mgz + \frac{p_z^2}{2m}$$

Solving for p_z and plugging into

$$J = \oint p\, dq$$

we get

$$J_z = \sqrt{2m} \oint (E_z - mgz)^{1/2}\, dz$$

we can do this closed integral by integrating from 0 to h and multiplying by 2.

$$J_z = 2\sqrt{2m}\frac{2}{3}(E_z - mgz)^{3/2}(\frac{-1}{mg})\bigg|_0^h$$

The original energy given to it in the z direction will be mgh, its potential energy when released from rest. Thus the first part of this evaluated integral is zero. Only the second part remains:

$$J_z = \frac{4}{3}\sqrt{2m}\frac{1}{mg}E_z^{3/2}$$

Solved in terms of E_z

$$E_z = (\frac{3}{4}g\sqrt{\frac{m}{2}}J_z)^{2/3}$$

The frequency is

$$\nu_z = \frac{\partial E_z}{\partial J_z} = (\frac{2}{3}(\frac{3}{4}g\sqrt{\frac{m}{2}})^{2/3})\frac{1}{J^{1/3}}$$

All we have to do now is plug what J_z is into this expression and simplify the algebra. As you may already see there are many different steps to take to simplify, I'll show one.

$$\nu_z = (\frac{2}{3}(\frac{3}{4}g\sqrt{\frac{m}{2}})^{2/3})\frac{1}{[\frac{4}{3g}\sqrt{\frac{2}{m}}(mgh)^{3/2}]^{1/3}}$$

Now we have a wonderful mess. Lets gather the numbers, and the constants to one side

$$\nu_z = \frac{\frac{2}{3}(\frac{3}{4})^{2/3}\frac{1}{2^{1/3}}}{(\frac{4}{3})^{1/3}2^{1/6}}\frac{g^{2/3}m^{1/3}g^{1/3}m^{1/6}}{m^{1/2}g^{1/2}h^{1/2}}$$

You may see, with some careful observation, that the m's cancel, and the constant part becomes

$$\frac{g^{1/2}}{h^{1/2}}$$

The number part simplifies down to

$$\frac{1}{2\sqrt{2}}$$

Thus we have

$$\nu_z = \frac{1}{2\sqrt{2}}\sqrt{\frac{g}{h}} = \frac{1}{2}\sqrt{\frac{g}{2h}}$$

as we were looking for from Newton's trivial method. Yay! Our two frequencies together

$$\nu_\theta = \frac{\dot{\theta}}{2\pi}$$

$$\nu_z = \frac{1}{2}\sqrt{\frac{g}{2h}}$$

The condition for the same path to be retraced is that the ratio of the frequencies to be a rational number. This is explained via closed Lissajous figures and two commensurate expressions at the bottom of page 462 in Goldstein.