(M,g) is a Rieannian manifold with Levi-Civita connection ∇。

ϕ:$\mathrm{M} \rightarrow \mathrm{M}$ is an isometry $\Leftrightarrow \phi^{*} g_{\phi(p)} = g_{p}$

Where the diffeomorphism ϕ is an isometry(等距同構)，preserve the metric。

$\varphi_{t}: M \rightarrow M$ is a one-parameter group of isometries，$X \in \chi(M)$

$X_{p} := \frac{d}{dt}\Big|_{t=0} \varphi_{t}(p)$ is called the Killing vector field associated to φ_{t}

$X, Y \in \chi(M)$，$L_{X} Y := \frac{d}{dt}((\varphi_{-t})_{*}\mathrm{Y})_{t=0}$，where $\{\varphi_{t}\}_{t \in I}$ is the flow of X。

And $L_{X} Y = [X, Y]$

$L_{X} \omega := \frac{d}{dt}(\varphi_{t}^{*} \omega)\Big|_{t=0}$

1. If X is a Killing field then $L_{X} g = 0$

   Manifold 的 metric 在這組向量的方向上保持不變。(preserves the metric)
   Flows generated by Killing fields are continuous isometries og the manifold。

   A vector field $K = K^{\mu} \partial_{\mu}$ on M is said to be a Killing vector field if the

   infinitesimal displacement $\varphi: x^{\mu} \rightarrow x^{\mu} + \varepsilon K^{\mu}$ generates an isometry。

   $$L_{V} g_{\mu\nu} = V^{\sigma} \nabla_{\sigma} g_{\mu\nu} + (\nabla_{\mu} V^{\lambda}) g_{\lambda\nu} + (\nabla_{\nu} V^{\lambda}) g_{\mu\lambda} = \nabla_{\mu} \nabla_{\nu} + \nabla_{\nu} \nabla_{\mu}$$

   Or $\mathrm{L}_{V} g_{\mu\nu} = 2\nabla(_{\mu} V_{\nu})$

2. Show that $X^{\kappa} \partial_{\kappa} g_{\mu\nu} + \partial_{\mu} X^{\kappa} g_{\kappa\nu} + \partial_{\nu} X^{\kappa} g_{\mu\kappa} = 0$ if X is a Killing vector field。And these are the so-called Killing equation。

3. $\nabla_{\mu} X^{\nu} + \nabla_{\nu} X^{\mu} = 0$ is the Killing equation

4. $< \nabla_{Y} X, Z > + < \nabla_{Z} X, Y > = 0$ for all $Y, Z \in \chi(M)$

5. If $c: I \to M$ is a geodesic，then $<\dot{c}(t), X_{c(t)}>$ is constant。

$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda$ the covariant derivative of a vector field V^ν。

Riemann tensor $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$

$R^{\ \ \ \rho}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\sigma\nu} - \partial_\nu \Gamma^\rho_{\sigma\mu} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\sigma\nu} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\sigma\mu}$

$R_{\rho\sigma\mu\nu} = g_{\rho\lambda} R^\lambda_{\sigma\mu\nu}$

$R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu}$ ， $R_{\rho\sigma\mu\nu} = -R_{\rho\sigma\nu\mu}$ ， $R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}$ ， $R_{\rho[\sigma\mu\nu]} = 0$

Bianchi Identity :

M is a manifold with symmetric connection ∇ ($\nabla_X Y - \nabla_Y X = [X,Y]$)，then

$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$

$R_{ijkl} + R_{iklj} + R_{iljk} = 0$

Ricci tensor $R_{\mu\nu} = R^\lambda_{\mu\lambda\nu}$

Scalar curvature $R = g^{\mu\nu} R_{\mu\nu}$

Exercises

1. Let ξ be a Killing vector field of constant length，prove that the integral curves of ξ are geodesics of (M ,g)
2. If ξ is Killing then $\nabla_X(\nabla\xi) = R(X,\xi)$ for all X
3. Show that any Killing vector K^μ satifies (1) $\nabla_\mu \nabla_\sigma K^\rho = R^\rho_{\sigma\mu\nu} K^\nu$ (2) $K^\lambda \nabla_\lambda R = 0$