One-parameter group

M is a smooth manifold

$\varphi: R\times M \to M \quad \varphi_t(p) := \varphi(t,p)$ satisfies：

1. $\varphi_0(p) = p$
2. $\varphi_s \circ \varphi_t = \varphi_{s+t}$ for all $s,t \in R$

Then $\{\varphi_t\}$ is called a one-parameter group onM

Examples

1. The flow of a vector field
2. $\varphi_A(t) = \exp(tA)$ is a one-parameter group with $\frac{d}{dt}\exp(tA)\big|_{t=0} = A$

   where $\exp A = \sum_{k=0}^{\infty} \frac{1}{k!}A^k$ defines a map $M_{n\times n} \xrightarrow{\exp} GL(n,R)$

   and $M_{n\times n}(R)$ is the Lie algebra of GL(n,R) with [A,B]=AB-BA

其他 DeepSeek 提供的 one-parameter groups

1. Modular group in number theory
2. Rotation groups

- The group of rotations in $\mathbb{R}^2$ (or $\mathbb{R}^3$) can be parametrized by an angle θ. For example:

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

  where $\theta \in \mathbb{R}$. This forms a one-parameter group under matrix multiplication.

3. Scaling froups

- The group of scaling transformations in $\mathbb{R}^n$ can be parametrized by a scaling factor $\lambda > 0$:

$$S_\lambda(x) = \lambda x.$$

  This forms a one-parameter group under composition.

4. Heisenberg group
5. Affine groups

- The affine group $\mathrm{Aff}(1)$ consists of transformations of the form:

$$f_t(x) = a_t x + b_t,$$

  where $a_t > 0$ and $b_t \in \mathbb{R}$. Specific choices of a_t and b_t can give rise to one-parameter subgroups.

6. Lie groups and their subgroups

- In general, any Lie group G has one-parameter subgroups corresponding to elements of its Lie algebra $\mathfrak{g}$. These are not limited to matrix groups and can include more abstract groups, such as diffeomorphism groups or gauge groups in physics.

7. Dynamics systems

- In dynamical systems, one-parameter groups can arise from the time evolution of a system. For example, the flow of a Hamiltonian system or the time evolution of a quantum state under a Hamiltonian operator can form a one-parameter group.

In classical mechanics，the flows of a Hamiltonian system is a one-parameter group of symplectomorphisms。

- ◦ The heat equation $\partial_t u = \Delta u$ generates a one-parameter semigroup of operators that describe the evolution of temperature distributions.
- ◦ The wave equation $\partial_{tt} u = \Delta u$ generates a one-parameter group of transformations on phase space.

- One-parameter groups describe geometric flows, such as:
  - The Ricci flow in differential geometry, which evolves a Riemannian metric over time.
  - Mean curvature flow, which describes the evolution of surfaces in Riemannian manifolds.

8. Translation groups

- The group of translations on $\mathbb{R}^n$ forms a one-parameter group when parametrized by a single direction. For example:

$$T_t(x) = x + tv,$$

where $v \in \mathbb{R}^n$ is a fixed vector and $t \in \mathbb{R}$.

9. Galilean and Lorentz boosts

- In physics, Galilean boosts (in non-relativistic mechanics) and Lorentz boosts (in special relativity) form one-parameter groups. For example, a Lorentz boost in the x-direction is parametrized by rapidity ϕ:

$$\Lambda_\phi = \begin{pmatrix} \cosh\phi & -\sinh\phi \\ -\sinh\phi & \cosh\phi \end{pmatrix}.$$

- In general relativity, one-parameter groups describe the flow of spacetime symmetries. For example:
  - Killing vector fields generate one-parameter groups of isometries, which preserve the metric of spacetime.
  - Cosmological models often use one-parameter groups to describe the expansion or contraction of the universe.

10. Circle actions

- The circle group S^1 (or $\mathbb{R}/\mathbb{Z}$) acts on spaces via periodic transformations. For example, the action of S^1 on $\mathbb{C}$ by multiplication:

$$z \mapsto e^{2\pi i t} z,$$

where $t \in \mathbb{R}$, forms a one-parameter group.

Q：在 M 上給定一個光滑的切向量場 X，是否存在 M 的 one-parameter group φ_t，使得 X 是φ_t 所誘導的切向量場？

A：若 M 是 compact differential manifold，X 是 M 上的 smooth 切向量場，則 X 在 M 上決定一個 one-parameter group。

Relationship between tagent vector fields and one-parameter groups：

- A smooth tangent vector field X can generate a **local one-parameter group of transformations** $\phi(t)$, which is constructed from the integral curves of X.
- Specifically, for each point $p \in M$, there exists an integral curve $\gamma_p(t)$ satisfying:

$$\gamma_p(0) = p \quad \text{and} \quad \frac{d}{dt}\gamma_p(t) = X(\gamma_p(t)).$$

- These integral curves locally define a one-parameter group $\phi(t)$, where $\phi(t)(p) = \gamma_p(t)$.

If the manifold is compact or the vector field X is complete (meaning its intehral curves are defined foe all time $t \in R$)，then the one-parameter group φ_t can be defined globally on M。

If M is non-compact and X is not complete，then φ_t may only exist locally。That is，for each point $p \in M$，there exists a neighborhood U and a time interval $(-\varepsilon, \varepsilon)$ such that φ_t is defined onU。

Induced Tangent Vector Field:

- The tangent vector field X induced by the one-parameter group $\phi(t)$ is defined as:

$$X_p = \frac{d}{dt}\phi(t)(p)\Big|_{t=0}.$$

- This means X is the **generator** of the one-parameter group $\phi(t)$.