What is a symplectic form？

A**symplectic form**is a fundamental concept in differential geometry and symplectic geometry。

It is a special type of differential 2-form that provides a geometric framework for studying **Hamiltonian mechanics**、dynamical systems，and other areas of mathematics and physics。

Asymplectic form on a smooth manifold M is a closed, non-degenerate 2-form ω。

Specifically：

1. **2-Form**: ω is a differential 2-form, meaning it assigns to each point $p \in M$ a skew-symmetric bilinear map:

$$\omega_p : T_pM \times T_pM \to \mathbb{R},$$

where T_pM is the tangent space to M at p.

2. **Closed**: The exterior derivative of ω vanishes, i.e.,

$$d\omega = 0.$$

This condition ensures that ω is locally exact (by the Poincaré lemma) and is related to the conservation of energy in Hamiltonian mechanics.

3. **Non-Degenerate**: For every nonzero tangent vector $v \in T_pM$, there exists another tangent vector $w \in T_pM$ such that:

$$\omega_p(v, w) \neq 0.$$

Non-degeneracy implies that ω induces an isomorphism between the tangent space T_pM and its dual space T_p^*M.

Key properties：

1. **Skew-Symmetry**: For all tangent vectors $X, Y \in T_pM$,

$$\omega(X, Y) = -\omega(Y, X).$$

This property distinguishes symplectic forms from Riemannian metrics, which are symmetric.

2. **Dimensionality**: A symplectic form can only exist on even-dimensional manifolds. If $\dim M = 2n$, then ω^n (the n-th exterior power of ω) is a volume form on M.
3. **Symplectic Manifold**: A manifold M equipped with a symplectic form ω is called a **symplectic manifold**.

On a Kahler manifold，the Kahler form ω is a symplectic form that is compatible with the complex structure and Riemannian metric。