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H.Poincare1904 R.Hamilton1982 G.Perelman2006

Poincare conjecture：

A closed，smooth，simply connected 3-manifold is diffeomorphic to S^3

[里奇流與 Poincare 猜想(1999 數學傳播 張樹城]

Ricci flow 指的就是這個偏微分方程：$\frac{\partial g(t)}{\partial t} = -2Ric(g(t))$

帥哥 Nick Sheridan 介紹 Ricci flow 文章中 p.38 的例子：

Given a smooth immersion $X_0 : R/Z \to R^2$

We can evolve it by taking $X : R/Z \times [0,T) \to R^2$ such that

$\begin{cases} \frac{\partial X}{\partial t}(u,t) = -\kappa N(u,t) \\ X(u,0) = X_0(u) \end{cases}$ $\cdots$(*)This is the curve-shortening flow equation。

Here κ is the curvature of our curve，N is the unit normal vector，defined by

$-\kappa N = \frac{\partial^2 X}{\partial s^2}$

Example：The shrinking circle

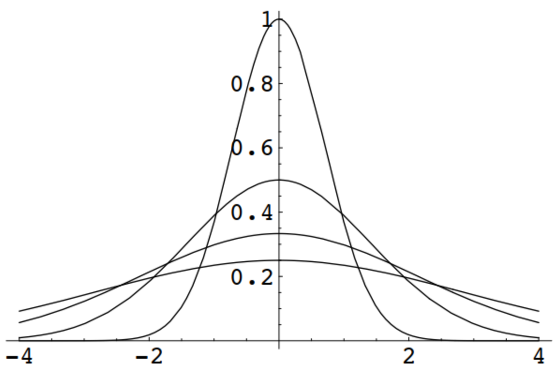
If our initial curve X_0 is a circle of radius r_0 centred at the origin，then the solution will take the form $X(u,t) = r(t)(\cos u, \sin u)$。

We have $N = \frac{X}{r}, \kappa = \frac{1}{r}$，so the curve-shortening equation becomes $\frac{dr}{dt} = -\frac{1}{r}$ which has the solution $X(u,t) = \sqrt{r_0^2 - 2t}(\cos u, \sin u)$

So the circle shrinks to a point at a finite tme $t = \frac{r_0}{\sqrt{2}}$

要了解以上的例子，我們先回顧一個簡單的 PDE：heat equation on R

(1) $u_t - ku_{xx} = 0$ The fundamental solution is $G(x,t) = \frac{1}{\sqrt{4\pi kt}} \exp(-\frac{x^2}{4kt})$



這個 fundamental solution 就是 Normal distribution， $\mu=0, \sigma=\sqrt{2\pi t}$

$X \sim N(\mu, \sigma^2)$

$f(x)=\frac{1}{\sigma\sqrt{2\pi}}\exp(-\frac{(x-\mu)^2}{2\sigma^2})$

Normal distribution 在物理學中稱為 Gaussian distribution。

(2) $\begin{cases} u_t - ku_{xx} = f(x,t), t>0 \\ u\big|_{t=0} = g(x) \end{cases}$

The particular solution is given by

$$u(x,t)=\int_{-\infty}^{\infty} G(x-y,t)g(y)dy+\int_0^t\int_{-\infty}^{\infty} G(x-y,t-s)f(y,s)dyds \text{ ...(**)}$$

$\begin{cases} u_t = ku_{xx}, t>0 \\ u\big|_{t=0} = x \end{cases}$ 解 $u(x,t)$

在 (**) 中，let f(x,t)=0，g(y)=y，則 $u(x,t)=\frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} e^{-\frac{(y-x)^2}{4kt}} y dy$

$$=\frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty}(y-x)e^{-\frac{(y-x)^2}{4kt}}dy+\frac{x}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty}e^{-\frac{(y-x)^2}{4kt}}dy = x$$

$\frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty}e^{-\frac{(y-x)^2}{4kt}}dy=1$ 這是 normal distribution $\int_{-\infty}^{\infty} f(x)dx=1$

前一項，奇函數積分 $\int_{-\infty}^{\infty} ue^{-\frac{u^2}{4kt}}du=0$ =0，後一項積分值=x。

然後回到 curve-shortening flow equation

$$\begin{cases} \frac{\partial X}{\partial t}(u,t)=\frac{\partial^2 X}{\partial s^2} & \text{...(*)} \\ X(u,0)=X_0(u)=r_0(\cos u, \sin u) \end{cases}$$

Then the solution of (*) is $X(u,t)=r(t)(\cos u, \sin u)$ ，where $r(0)=r_0$ 。

把 PDE 的熱傳導方程，常態分布，一個 manifold(此處是一個圓)隨一個 flow(此處是 curve-shortening flow)演進成一個點一起看，在枯寂無聊的歲月總算有點樂趣。