§ transport equation

$u_t + cu_x = 0$ ... (1)

$$\begin{cases} u_t + cu_x = 0, x \in R, t > 0 \\ u\big|_{t=0} = f(x), x \in R \end{cases} \quad \text{... (2)}$$

1. The general solution of (1) is $u(t,x) = \phi(x - ct)$
2. The particular solution of (2) is $u(t,x) = f(x - ct)$

Method of charateritics

Example $u_t + 2u_x = 0, u(0,x) = \dfrac{1}{1+x^2}$ $\qquad u(t,x) = \dfrac{1}{1+(x-2t)^2}$

§ heat equatuon

(1) $u_t - ku_{xx} = 0$ The fundamental solution is $G(x,t) = \dfrac{1}{\sqrt{4\pi kt}} e^{\frac{-x^2}{4kt}}$

(2) $\begin{cases} u_t - ku_{xx} = f(x,t), t > 0 \\ u\big|_{t=0} = g(x) \end{cases}$

The constant k is called the thermal diffusivity(熱擴散率)。
The particular solution is given by

$$u(x,t) = \int_{-\infty}^{\infty} G(x-y,t) g(y) dy + \int_0^t \int_{-\infty}^{\infty} G(x-y,t-s) f(y,s) dy ds$$

Separation method

1. $\begin{cases} u_t = ku_{xx} \\ u\big|_{t=0} = x^2 \end{cases}, t > 0$

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} y^2 e^{-\frac{-(y-x)^2}{4kt}} dy$$

Let $p = \dfrac{y-x}{\sqrt{4kt}}, dp = \dfrac{dy}{\sqrt{4kt}}$ note that $\dfrac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2} dp = 1, \int_{-\infty}^{\infty} p e^{-p^2} dp = 0$

Then $u(x,t) = \dfrac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} (x + \sqrt{4kt}\, p)^2 e^{-p^2} dp = \ldots = x^2 + 2kt$

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

§ 3 wave equation

(1) 齊次 $u_{tt} - c^2 u_{xx} = 0$

(2) $\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t), x \in R, t > 0 \\ u\big|_{t=0} = g(x), x \in R \\ u_t\big|_{t=0} = h(x), x \in R \end{cases}$

(1) 的一般解 $u(t,x) = \phi(x-ct) + \psi(x+ct)$

Brook Taylor 1714 年

(2) 的特別解

$$u(t,x) = \frac{g(x+ct)+g(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} h(s)ds + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)dyds$$

§ 變數分離法

1. $\begin{cases} u_{tt} = c^2 u_{xx}, 0 < x < l, t > 0 \\ u(0,t) = u(l,t) = 0, t \ge 0 \\ u(x,0) = f(x), u_t(x,0) = g(x), 0 \le x \le l \end{cases}$