Killing vector fields on S^3

Killing vector field on S^3 are vector fields that generate isometries of S^3。
(They preserve the metric，$L_K g = 0$)

Killing vector fields on S^3：

1. **Isometry Group**: The isometry group of S^3 with the round metric is $SO(4)$, the group of orthogonal transformations of $\mathbb{R}^4$ with determinant 1. This group has dimension 6, so there are 6 linearly independent Killing vector fields on S^3.

Killing vector fields on S^3 are the generators of the isometry group SO(4)

2. **Lie Algebra Structure**: The Killing vector fields on S^3 form a Lie algebra isomorphic to the Lie algebra $\mathfrak{so}(4)$, which is the Lie algebra of $SO(4)$. The Lie algebra $\mathfrak{so}(4)$ is 6-dimensional and can be decomposed as $\mathfrak{so}(4) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$.

3. **Explicit Construction**: The Killing vector fields on S^3 can be explicitly constructed using the embedding of S^3 in $\mathbb{R}^4$. Let (x_1, x_2, x_3, x_4) be coordinates in $\mathbb{R}^4$, so that S^3 is defined by $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$. The Killing vector fields correspond to the infinitesimal generators of rotations in $\mathbb{R}^4$. These can be written as:

$$X_{ij} = x_i \partial_j - x_j \partial_i,$$

where $i, j = 1, 2, 3, 4$ and $i < j$. There are 6 such vector fields, corresponding to the 6 independent planes of rotation in $\mathbb{R}^4$.

There are 6 linearly independent Killing vector fields，corresponding to the 6 dimensions of SO(4)

4. **Relation to $SU(2)$**: The 3-sphere S^3 can be identified with the Lie group $SU(2)$, and the Killing vector fields correspond to the left-invariant vector fields on $SU(2)$. These vector fields are generated by the Lie algebra $\mathfrak{su}(2)$, which is 3-dimensional. However, the full set of Killing vector fields on S^3 includes both left-invariant and right-invariant vector fields, giving a total of 6 independent Killing vector fields.