§ Euler-Lagrange flows

$L: TM \to R$ Lagrangian，$u:[0,1] \to M$

$$A(u) = \int_0^1 L(u(t), \dot{u}(t))dt$$

$$\frac{\partial L}{\partial u}(u, \dot{u}) - \frac{d}{dt}(\frac{\partial L}{\partial v}(u, \dot{u}) = 0$$

If M is compact，the extremals of A give rise to a complete flow $\phi_t : TM \to TM$ called the Euler-Lagrange flow of the Lagrangian。

[DG12]Geometric Analysis p.23

Lemma

The Euler-Lagrange equations for the energy E are

$$\ddot{x}^i(t) + \Gamma^i_{jk}(x(t))\dot{x}^j(t)\dot{x}^k(t) = 0$$

Where $\Gamma^i_{jk} = \frac{1}{2} g^{il}(g_{jl,k} + g_{kl,j} - g_{jk,l})$

1. For $I(x) = \int_a^b f(x(t), \dot{x}(t), t)dt$ then the E-L equation are $\frac{\partial f}{\partial x^i} - \frac{d}{dt}(\frac{\partial f}{\partial \dot{x}^i}) = 0$

2. $E(\gamma) = \frac{1}{2}\int g_{jk}(x(t))\dot{x}^j \dot{x}^k dt$

Note that $\frac{\partial}{\partial \dot{x}^i}(g_{jk}\dot{x}^j \dot{x}^k) = g_{ik}\dot{x}^k + g_{ji}\dot{x}^j$ 因為 $\frac{\partial}{\partial \dot{x}^i}\dot{x}^j = \delta_{ij}$

The E-L equations are $\frac{\partial}{\partial x^i}(g_{jk}\dot{x}^j \dot{x}^k) - \frac{d}{dt}(\frac{\partial}{\partial \dot{x}^i} g_{jk}\dot{x}^j \dot{x}^k) = 0$

$$\frac{\partial g_{jk}}{\partial x^i}\dot{x}^j \dot{x}^k - \left\{ \frac{\partial g_{ik}}{\partial x^l}\frac{\partial x^l}{\partial t}\dot{x}^k + \frac{\partial g_{jl}}{\partial x^l}\frac{\partial x^l}{\partial t}\dot{x}^j + g_{ik}\ddot{x}^k + g_{ji}\ddot{x}^j \right\} = 0$$

$$g_{jk,i}\dot{x}^j \dot{x}^k - \left\{ g_{ik,l}\dot{x}^l \dot{x}^k + g_{ji,l}\dot{x}^l \dot{x}^j + g_{ik}\ddot{x}^k + g_{ji}\ddot{x}^j \right\} = 0$$

Renaming some indices and using the symmetry $g_{ik} = g_{ki}$, we get

$$2g_{\ell m}\ddot{x}^m + (g_{\ell k,j} + g_{j\ell,k} - g_{jk,\ell})\dot{x}^j \dot{x}^k = 0, \quad \ell = 1, \ldots, d,$$

And from this $g^{il} g_{lm} \ddot{x}^m + \frac{1}{2} g^{il} (g_{lk,j} + g_{jl,k} - g_{jk,l}) \dot{x}^j \dot{x}^k = 0$

$g^{il} g_{lm} = \delta_{im}$ ，so $g^{il} g_{lm} \ddot{x}^m = \ddot{x}^i$ ，that is $\ddot{x}^i + \Gamma^i_{jk} \dot{x}^j \dot{x}^j = 0$

§ Cartan magic formula $L_X \omega = \iota_X d\omega + d(\iota_X \omega)$

右式前者是 interior product(interior derivatve)，後者是 exterior derivative

$X \in \chi(M)$ ，$L_X \omega = \frac{d}{dt} (\varphi_t^* \omega)\big|_{t=0}$

$L_X d\omega = dL_X \omega$

$L_X (\omega \wedge \eta) = L_X \omega \wedge \eta + \omega \wedge L_X \eta$

1. 例子 in RG1102

$X = F \frac{\partial}{\partial x} + G \frac{\partial}{\partial y} + H \frac{\partial}{\partial z}$ ，volume form $dv = dx \wedge dy \wedge dz$

$L_X dx = d(L_X x) = d(Xx) = dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy + \frac{\partial F}{\partial z} dz$ ，Then

$L_X dv = L_X (dx \wedge dy \wedge dz) = (L_X dx) \wedge dy \wedge dz + dx \wedge (L_X dy) \wedge dz + dx \wedge dy \wedge (L_X dz)$

$= (\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}) dv = (divX) dv$

驗證 所謂的 interior product 如何運算。

$\iota_X (dx_1 \wedge dx_2 ... \wedge dx_n) = \sum_{r=1}^{n} (-1)^{r-1} f_r dx_1 \wedge ... \wedge \hat{dx_r} \wedge ... \wedge dx_n$

這裡 $\hat{dx_r}$ 表示把 dx_r 省略

所以 $X = F \frac{\partial}{\partial x} + G \frac{\partial}{\partial y} + H \frac{\partial}{\partial z} = (F, G, H)$ ，$\omega = dx \wedge dy \wedge dz$

$\iota_X \omega = F dy \wedge dz - G dx \wedge dz + H dx \wedge dy$

$d(\iota_X \omega) = dF \wedge dy \wedge dz - dG \wedge dx \wedge dz + dH \wedge dx \wedge dy$

其中 $dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy + \frac{\partial F}{\partial z} dz$ ，...

所以 $d(\iota_X \omega) = (\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} + \frac{\partial H}{\partial z}) dx \wedge dy \wedge dz$