§ V.I.Arnold 1937~2010 寫了不少書

1. Mathematical Methods of Classical Mechanics
2. Ordinary Differential Equations
3. Lectures on PDE
4. Mathematical Understanding of Nature：
5. Geometrical Methods in the theory of ODE

§ E-L equation for

1. 旋轉體表面積

$$S(y)=2\pi\int_{x_0}^{x_1} y\sqrt{1+(y')^2}\,dx \text{，} \quad y=c\sqrt{1+(y')^2} \Rightarrow y=c\cosh\left(\frac{x+c'}{c}\right)$$

2. 最速降線

$$T=\int_{x_1}^{x_2}\sqrt{\frac{1+(y')^2}{2gy}}\,dx \text{，} \quad y+y(y')^2=c \Rightarrow \begin{cases} x=k(\theta-\sin\theta) \\ y=k(1-\cos\theta) \end{cases}$$

這裡有兩個 ODE 要解

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古典力學　　幾何力學　　廣義相對論

ODE　　$\Rightarrow$PDE　　$\Rightarrow$　　　　$\Rightarrow$Quantum Gravity

微分幾何　　Lie groups　　量子力學

變分法

§ 幾何流線

Geodesic flows　　Gabriel P. Paternain

M is a complete Riemannian manifold

$\gamma_{(x,v)}(t)$ is the unique geodesic with $\gamma_{(x,v)}(0)=x, \dot{\gamma}_{(x,v)}(0)=v$

$\varphi_t : TM \to TM$ a diffeomorphism $\varphi_t(x,v) := (\gamma_{(x,v)}(t), \dot{\lambda}_{(x,v)}(t))$

Then $\{\varphi_t\}$ is a flow with $\varphi_{t+s}=\varphi_t \circ \varphi_s$

Let SM be the unit tangent bundle of M，$|v|=1$

Because geodesic travel with constant speed ，φ_t leaves SM invariant。That is ，given $(x,v)\in SM$ ，$\varphi_t(x,v)\in SM$ for all t

Euler-Lagrange flows

Ricci flows

Phase space：position，velocity

Find phase flows of some DE

例 (1) $\dot{x}=1$ (2) $\dot{x}=x-1$ (3) $\dot{x}=\sin x, 0<x<\pi$

ANS 1. $g^t=x+t$ 2. $g^t=(x-1)e^t+1$ 3. $g^t=2\operatorname{arccot}(e^{-t}\cot\frac{x}{2})$

例 (1) $\begin{cases}\dot{x}=y\\ \dot{y}=1\end{cases}$ (2) $\begin{cases}\dot{x}=\sin y\\ \dot{y}=0\end{cases}$

ANS 1. $g^t=(x+ty+\frac{1}{2}t^2, y+t)$ 2. $g^t=(x+t\sin y, y)$

§ 關於 geodesic $\ddot{u}^i+\Gamma^i_{jk}\dot{u}^j\dot{u}^k=0$

A geodesic is a curve along which the tangent vector is parallel transported。

1. 變分法 [大域] 弧長的變分 能量的變分(Synge-Weistein theorem)
2. Minimal-coupling principle(對偶原理)
3. 等價原理：重力場決定伽利略時空的曲線 這些曲線是 Cartan connection 的測地線。
4. Geodesic deviation equation (Jacobi field 是特例)
5. Heat flow
6. Geometric Analysis 中用能量定義 geodesic。$E(\gamma)=\frac{1}{2}\int_a^b\left|\frac{d\gamma(t)}{dt}\right|^2 dt$，then E-L equation for E。 K.Uhlenbeck 用能量定義 minimal surface 從而發展到 gauge theory。
7. 古典的方法

§ A symplectic manifold (M^{2n},ω)

M a smooth manifold

2-form ω： (1)smooth (2closed (3)nondegenerated

$H:M\to R$ a Hamiltonian(哈密頓符)

ξ_H Hamiltonian vector field of H $dH(v)=\omega(v,\xi_H)$

Hamiltonian flow:=the flow of ξ_H

則 Hamiltonian flow 與 symplectic form 有很好的關係：$(g^t)^*\omega=\omega$

F、G 是 symplectic manifold (M,ω) 上的 Hamitonuan

定義 Poisson bracket $\{F,G\}:=dF(\xi_G)$ F 沿 G 的 Hamilton flow 方向的導數
滿足

1. Linear
2. Skew-symmetric
3. 滿足 Jacobi identity

兩函數 f, g 稱為對合的(involute)$\Leftrightarrow \{f,g\}=0$

A function f on M is an integral for the Hamiltonian flow of H $\Leftrightarrow$ f and H are in involution。

Symplectic manifold (M,ω) 上、Hamiltonian H 的 flow 稱為可積的(or completely integrable) 若存在 n 個獨立的 integral $f_1=H, f_2,,,, f_n$ in involution

§ 習作

1. Sean Carrell 的 Spacetime and Geometry 第三章 曲率
2. Geometric Analysis 第一章
3. Hyperbolic plane 後面有 4 個習作

§ Questions

1. 何謂 left invariant vector fields
2.