[Div-Grad-Curl-and-All-That]第四章 Ex 15 p.149

f 是可微的位置函數，稱$\nabla^2 f = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$為波動方程。(PDE 中寫成$u_{tt} - c^2 u_{xx} = 0$)

利用 Maxwell 方程證明電荷與電流真空的空間(即ρ, J 皆為零)，在直角坐標系中電場 E、磁場 B 的三個分量滿足波速 v=c 的波動方程。例如$\nabla^2 E_x = \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2}$

因此 電磁波在真空中以光速前進是 Maxwell 方程的一個結論。

Maxwell Equations (James Clerk Maxwell 1831~1879)

$\nabla \cdot \mathrm{E} = 4\pi\rho$

$\nabla \times E = -\frac{1}{c}\frac{\partial B}{\partial t}$

$\nabla \cdot B = 0$

$\nabla \times B = \frac{1}{c}(4\pi J + \frac{\partial E}{\partial t})$

要先證明

1. $\frac{\partial}{\partial t}(\nabla \times A) = \nabla \times (\frac{\partial}{\partial t} A)$
2. $\nabla \times (\nabla \times A) = \nabla(\nabla \cdot A) - \nabla^2 A$

$\because J = 0$ ，$\nabla \times B = \frac{1}{c}\frac{\partial E}{\partial t}$

$$\frac{1}{c}\frac{\partial^2 E}{\partial t^2} = \frac{\partial}{\partial t}(\nabla \times B) = \nabla \times (\frac{\partial}{\partial t} B) = -c\nabla \times (\nabla \times E) = -c\left\{\nabla(\nabla \cdot E) - \nabla^2 E\right\} = c\nabla^2 E$$

所以$\nabla^2 E = \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2}$

同理 $\nabla \times E = -\frac{1}{c}\frac{\partial B}{\partial t}$

$$-\frac{1}{c}\frac{\partial^2 B}{\partial t^2} = \frac{\partial}{\partial t}(\nabla \times E) = \nabla \times (\frac{\partial}{\partial t} E) = c\nabla \times (\nabla \times B) = c\left\{\nabla(\nabla \cdot B) - \nabla^2 B\right\} = -c\nabla^2 B$$

所以$\nabla^2 B = \frac{1}{c^2}\frac{\partial^2 B}{\partial t^2}$

以下驗證$\nabla\times(\nabla\times A)=\nabla(\nabla\cdot A)-\nabla^2 A$...(*)

假設 A=(P,Q,R)， $\nabla=(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}),\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$

只檢查 x 分量

$$\nabla\times(\nabla\times A)=\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ R_y-Q_z & P_z-R_x & Q_x-P_y \end{vmatrix}=(Q_{xy}-P_{yy}-P_{zz}+R_{xz},.,.)$$

$$\nabla(\nabla\cdot A)=\nabla(P_x+Q_y+\mathrm{R}_z)=(P_{xx}+Q_{yx}+R_{zx},.,.)$$

$\nabla^2 A=(P_{xx}+P_{yy}+P_{zz},.,.)$ ，(*)左右兩式的 x 分量相等，同理可得 y,z 分量，得證

或許有較好的方法。