III–20 Maxwell's equations are

$$\nabla \cdot \mathbf{E} = \rho/\epsilon_0, \qquad \nabla \cdot \mathbf{B} = 0,$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \qquad \text{and} \qquad \nabla \times \mathbf{B} = \epsilon_0\mu_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{J},$$

where **E** is the electric field, **B** the magnetic field, ρ the charge density, and **J** the current density. Use Maxwell's equations to derive the continuity equation

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0.$$

Interpret this equation.

Maxwell 方程式改成

$\nabla \cdot E = 4\pi\rho$　　　　$\nabla \cdot \mathrm{B} = ($

$\nabla \times E = -\frac{1}{c}\frac{\partial B}{\partial t}$　　　　$\nabla \times B = \frac{1}{c}\frac{\partial E}{\partial t} + \frac{4\pi}{c} J$

$$\nabla \cdot (\frac{\partial E}{\partial t}) = \frac{\partial}{\partial t}(\nabla \cdot E)$$

$$\nabla \cdot (\nabla \times B) = 0$$

$$\frac{1}{c}\frac{\partial}{\partial t}(\nabla \cdot E) + \frac{4\pi}{c}\nabla \cdot J = 0$$

$\frac{4\pi}{c}\frac{\partial \rho}{\partial t} + \frac{4\pi}{c}\nabla \cdot J = 0$，所以$\nabla \cdot J + \frac{\partial \rho}{\partial t} = 0$

另一證法

density $\rho(x, y, z, t)$，velocity v(x,y,z,t)，current density $J = \rho v$

If there are no sources or sinks prove that $\nabla \cdot J + \frac{\partial \rho}{\partial t} = 0$

At any time the mass of fluid within V is

$$M = \iiint_V \rho dV$$

$$\frac{\partial M}{\partial t} = \frac{\partial}{\partial t}\iiint_V \rho dV = \iiint_V \frac{\partial \rho}{\partial t} dV$$

The mass of fluid per unit time leaving V is $\iint_S \rho v \cdot ndS$

And the time rate of increase in mass is $-\iint_S \rho v \cdot ndS = -\iiint_V \nabla \cdot (\rho v) dV$ (divergence)

So $\iiint_V \frac{\partial \rho}{\partial t} dV = -\iint_S \rho v \cdot n dS = -\iiint_V \nabla \cdot (\rho v) dV$

$\iiint_V (\nabla \cdot (\rho v) + \frac{\partial \rho}{\partial t}) dV = 0$ ， $\rho v = J$

所以$\nabla \cdot J + \frac{\partial \rho}{\partial t} = 0$