II–7 Find the moment of inertia about the z-axis of the hemispherical shell of Problem II–6.

球殼以 z 軸為旋轉軸時的角慣量。 $I = \iiint\limits_D (x^2+y^2)\rho(x,y,z)dV$

Where $\rho = \dfrac{\rho_0}{R^2}(x^2+y^2)$ ， $z = f(x,y) = (R^2-x^2-y^2)^{\frac{1}{2}}$

$$I = \iint_S \frac{\rho_0}{R}\frac{(x^2+y^2)^2}{\sqrt{R^2-x^2-y^2}}dxdy$$

查表 $\int \sin^n x dx = -\dfrac{1}{n}\sin^{n-1} x\cos x + \dfrac{n-1}{n}\int \sin^{n-2} x dx$

先算一下 $\int \sin^5 x dx = -\dfrac{1}{5}\sin^4 x\cos x + \dfrac{4}{5}\int \sin^3 x dx$ ，

$$\int \sin^3 x dx = -\frac{1}{3}\sin^2 x\cos x - \frac{2}{3}\cos x$$

Let $r = R\sin\varphi, dr = R\cos\varphi d\varphi$ ， $\sqrt{R^2-r^2} = R\cos\varphi$

$$\int_0^R \frac{r^5}{\sqrt{R^2-r^2}}dr = R^5\int_0^{\frac{\pi}{2}}\sin^5\varphi d\varphi = \ldots = \frac{8}{15}R^5$$

$$I = \frac{\rho_0}{R}\int_0^{2\pi}\int_0^R \frac{r^5}{\sqrt{R^2-r^2}}drd\theta = \frac{\rho_0}{R}\times 2\pi\times\frac{8R^5}{15} = \frac{16\pi R^4\rho_0}{15}$$