II–6 The distribution of mass on the hemispherical shell

$$z = (R^2 - x^2 - y^2)^{1/2}$$

is given by

$$\sigma(x, y, z) = (\sigma_0/R^2)(x^2 + y^2).$$

where σ_0 is a constant. Find an expression in terms of σ_0 and R for the total mass of the shell.

$\rho(x, y, z)$ 稱為面密度(質量分布)函數，用 R，σ_0 表示球殼質量。

$dM = \rho dS$ ，

$$M = \iint_S \rho dS = \iint_R \rho \sqrt{1 + (\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2} dxdy = \iint_R \frac{\rho_0}{R^2}(x^2 + y^2)\frac{\mathrm{R}}{\sqrt{\mathrm{R}^2 - x^2 - y^2}} dxdy \ ,$$

where $z = f(x, y) = (R^2 - x^2 - y^2)^{\frac{1}{2}}$

$$= \frac{\rho_0}{R}\iint_R \frac{x^2 + y^2}{\sqrt{R^2 - x^2 - y^2}} dxdy$$ let $x = r\cos\theta, y = r\sin\theta$

$$= \frac{\rho_0}{R}\int_0^{2\pi}\int_0^R \frac{r^3}{\sqrt{R^2 - r^2}} drd\theta$$ let $r = R\sin\varphi, dr = R\cos\varphi d\varphi$

$$\int_0^R \frac{r^3 dr}{\sqrt{R^2 - r^2}} = \int_0^{\frac{\pi}{2}} \frac{R^4 \sin^3\varphi \cos\varphi d\varphi}{R\cos\varphi} = R^3 \int_0^{\frac{\pi}{2}} \sin^3\varphi d\varphi$$

$$\int \sin^3 x dx = \int \sin x(1 - \cos^2 x) dx = \int \sin x dx - \int \sin x \cos^2 x dx = -\cos x + \frac{1}{3}\cos^3 x$$

$$\int_0^{\frac{\pi}{2}} \sin^3\varphi d\varphi = -\cos\varphi \Big|_0^{\frac{\pi}{2}} - \frac{1}{3}u^3 \Big|_0^1 = \frac{2}{3}$$

$$R^3 \times \frac{\rho_0}{R}\int_0^{2\pi} \frac{2}{3} d\theta = \frac{4\pi R^2 \rho_0}{3}$$ This is the answer!