II-4

$$\iint_S G(x,y,z)dS = \iint_R G[x,y,\ f(x,y)]\cdot\sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}\,dxdy$$

In each of the following，evaluate the surface integral $\iint_S G(x,y,z)dS$

(a) G(x,y,z)=z，where S is the portion of the plane x+y+z=1 in the first octant。

(b) $G(x,y,z)=\dfrac{1}{1+4(x^2+y^2)}$ ，where S is the portion of the paraboloid $z=x^2+y^2$ between z=0 and z=1

(c) $G(x,y,z)=(1-x^2-y^2)^{\frac{3}{2}}$ ，where S is the hemispere $z=(1-x^2-y^2)^{\frac{1}{2}}$

(a) z=f(x,y)=1-x-y， $\sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}=\sqrt{3}$

$\iint_S zdS=\iint_R(1-x-y)\sqrt{3}dxdy$ ， $\iint_R(1-x-y)dxdy=\int_0^1\int_0^{1-y}(1-x-y)dxdy=\dfrac{1}{6}$

所以 $\iint_S G(x,y,z)dS=\dfrac{\sqrt{3}}{6}$

Surface unit normal= $\dfrac{1}{\sqrt{3}}(1,1,1)$ ，then $F=(0,0,\sqrt{3}z)$ ， $divF=\sqrt{3}$

$\iint_S zdS=\iint_S F\cdot\vec{n}dS=\iiint_V divFdV=\dfrac{\sqrt{3}}{6}$

(b) $z=f(x,y)=x^2+y^2$

$$\iint_S G(x,y,z)dS=\iint_R\frac{1}{1+4(x^2+y^2)}\sqrt{1+4(x^2+y^2)}dxdy=\iint_R\frac{1}{\sqrt{1+4(x^2+y^2)}}dxdy$$

Let $x=r\cos\theta, y=r\sin\theta$ ， $\iint_R\dfrac{1}{\sqrt{1+4(x^2+y^2)}}dxdy=\int_0^{2\pi}\int_0^1\dfrac{r}{\sqrt{1+4r^2}}drd\theta$

Let $r=\dfrac{1}{2}\tan\varphi$ ，then $dr=\dfrac{1}{2}\sec^2\varphi d\varphi$ ，

$$\int_0^1\frac{r}{\sqrt{1+4r^2}}dr=\int_0^{\tan^{-1}2}\frac{1}{4}\tan\varphi\sec\varphi d\varphi=\frac{1}{4}\sec\varphi\Big|_0^{\tan^{-1}2}=\frac{1}{4}(\sqrt{5}-1)$$

And $\iint_S G(x,y,z)dS=\dfrac{\pi}{2}(\sqrt{5}-1)$

(c) It's easy to check that $\iint_S G(x,y,z)dS = \iint_R (1-x^2-y^2)dxdy = \frac{\pi}{2}$