JACOBI’S THEOREM IN LORENTZIAN GEOMETRY

BY

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Abstract. We generalized the Jacobi’s theorem of Euclidean space to Minkowski space. Let \( c(s) \) be a timelike curve with arclength parameter \( s \) in the Minkowski space. Let \( \tau \) be the image of the Gauss map of the unit principal vector field of \( c(s) \) into the de Sitter space \( S^2_1 \). Assume that when the parameter \( s \) varies from \( s = 0 \) to \( s = a \), the image \( \tau \) is a simple closed curve and not null-homotopic to \( S^2 \) and \( k(0) = k(a) \) and \( w(0) = w(a) \), where \( k(t) \) and \( w(t) \) be a curvature and torsion at a point \( s = t \), respectively. Then \( \tau \) divides a “segment” of the de Sitter space into two regions with equal areas.

1. Introduction.

The Jacobi’s theorem in 3-dimensional Euclidean space \( \mathbb{E}^3 \) is followng:

**Theorem 1.1.**[3]. Let \( \alpha(s) : I \to \mathbb{E}^3 \) be a closed, regular, parametrized curve with nonzero curvature. Assume that the Gauss map \( \overline{\alpha} \) of the normal vector of \( \alpha(s) \) is simple in the unit sphere \( S^2 \). Then \( \overline{\alpha} \) divides \( S^2 \) into two regions with equal areas.

In the present paper we shall examine this theorem in 3-dimensional Minkowski space \( L^3 \). First we recall the Gauss-Bonnet theorem in a 2-
dimensional Lorentzian manifold, for this is the key theorem to give the Jacobi’s theorem. In section 3, we recall the “frame” of a curve in Lorentzian geometry, for the frame formulas in Lorentzian geometry are more complicated than that of Euclidean geometry. In section 4, we give the Jacobi’s theorem on a timelike curve (the theorem on a spacelike curve is almost the same as that of Euclidean case). The last section is devoted to give the Jacobi’s theorem on a null curve.

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2. Preliminaries. First we recall some definitions and the Gauss-Bonnet theorem for a domain in a 2-dimensional Lorentzian manifold. Since this section is devoted for the preliminary of our new theorems, the statement is abridged slightly. For full explanation about topics of this section, see [1], [2], [5], [7].

Let \( M \) be a Lorentzian manifold with the Lorentzian metric \( g \). A vector \( X \) at a point of \( M \) is called spacelike, timelike or null if \( g(X, X) > 0 \) or \( X = 0 \), \( g(X, X) < 0 \), \( g(X, X) = 0 \) and \( X \neq 0 \), respectively. The norm \( ||X|| \) of \( X \) is defined as \( ||X|| := \sqrt{|g(X, X)|} \). The complex-valued norm \( \langle X \rangle \) of \( X \) is defined as \( \langle X \rangle := \sqrt{g(X, X)} \), that is, \( \langle X \rangle \in \mathbb{R}^+ \cup \{0\} \cup \mathbb{R}^+i \), where \( \mathbb{R}^+ \) denotes the set of all positive numbers and \( i = \sqrt{-1} \).

On the 3-dimensional Minkowski space \( L^3 \), for any two arbitrary vectors \( X = (x_1, x_2, x_3) \) and \( Y = (y_1, y_2, y_3) \), \( g(X, Y) \) can be written as

\[
(2.1) \quad g(X, Y) = x_1y_1 + x_2y_2 - x_3y_3,
\]

that is, \( g \) is the inner product. The exterior product \( X \times Y \) is defined by

\[
(2.2) \quad X \times Y = (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, -(x_1y_2 - x_2y_1)).
\]
In $\mathbb{L}^3$, the de Sitter space $S^2_1$ is defined by setting

$$S^2_1 = \{ X \mid X \in \mathbb{L}^3, \quad g(X, X) = 1 \}.$$ 

For two non-null vectors $X$ and $Y$, non-directed sectional mesure $\varnothing = \varnothing(X, Y)$ is a complex number satisfying the equation

$$\cos \varnothing = \frac{g(X, Y)}{\langle X \rangle \cdot \langle Y \rangle} 
 \text{and defined as follows:}$$

(1) If

$$\frac{g(X, Y)}{\langle X \rangle \cdot \langle Y \rangle} \in [-1, 1],$$

then $\varnothing \in [0, \pi].$

(2) If

$$\frac{g(X, Y)}{\langle X \rangle \cdot \langle Y \rangle} > 1,$$

then $\varnothing = \theta i$ (when $\|X\| > 0, \|Y\| > 0$) or $\varnothing = \theta / i$ (when $\|X\| < 0, \|Y\| < 0$) is uniquely determined by (2.3).

(3) If

$$\frac{g(X, Y)}{\langle X \rangle \cdot \langle Y \rangle} < -1,$$

then $\varnothing = \pi - i \theta$ (when $\|X\| > 0, \|Y\| > 0$) or $\varnothing = \pi - \theta / i$ (when $\|X\| < 0, \|Y\| < 0$), where $\theta > 0$ is uniquely determined by (2.3).

(4) If

$$\frac{g(X, Y)}{\langle X \rangle \cdot \langle Y \rangle} \in \mathbb{R}i,$$

then $\varnothing = \frac{\pi}{2} + i \nu$, where $\nu$ s uniquely determined by (2.3).

In the Euclidean 2-space $\mathbb{R}^2$, we write a circle $S^1$ and give 4 arcs $ARC_0 := A_0A_1$, $ARC_1 := A_1A_2$, $ARC_2 := A_2A_3$, $ARC_3 := A_3A_4$, where $A_0 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), \ldots$
\[ A_1 = (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), \quad A_2 = (-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}), \quad A_3 = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) \] (where arcs do not include their end points). In \( \mathbb{R}^2 \), we define the Lorentzian metric \( g = (+, -) \) and make \( \mathbb{R}^2 \) to \( \mathbb{L}^2 \).

Then, for \( P, Q \in S^1 \), fundamental angle \( \angle POQ \) is defined as follows.

1. If \( P, Q \in ARC_j(j = \{0, 1, 2, 3\}) \), where \( \{0, 1, 2, 3\} \) denotes the quotient group modulo 4 of the natural number), then the fundamental angle \( \angle POQ \) is non-directed sectional measure \( \varnothing = \varnothing(\overrightarrow{OP}, \overrightarrow{OQ}) \).

2. If \( P \in ARC_j \) and \( Q \in ARC_{j+1} \), or \( Q \in ARC_j \) and \( P \in ARC_{j+1} \) and \( g(\overrightarrow{OP}, \overrightarrow{OQ}) = 0 \), then the fundamental angle \( \angle POQ \) is \( \frac{\pi}{2} \).

Next, we define the directed sectional measure as follows. When an angle \( \angle POQ \) is the fundamental angle, if the varying point moving from the initial point \( P \) to the terminal point \( Q \) along \( S^1 \) in the counterclockwise direction, then the directed sectional measure of the fundamental angle is defined to be the product of the fundamental angle by \( +1 \). If the varying point moves in the clockwise direction, product \( -1 \). When an angle \( \angle POQ \) is not the fundamental angle, we split the angle \( \angle POQ \) into successive non-overlapping fundamental angles. Then the directed sectional measure of \( \angle POQ \) is the summing up of fundamental angles. (We can easily see that the definitions of the directed sectional measure of “general” angle \( \angle POQ \) is independent of the choice of splittings).

Next, we shall define the geodesic curvature. Let \( M^2 \) be a 2-dimensional Lorentzian manifold with the Lorentzian metric \( g \). Suppose \( c = c(t) \) be a smooth curve on \( M^2 \). The length of \( c \) with respect \( (\cdot) \) from \( t = a \) to \( t = b \) is

\[ \alpha = \int_a^b \langle \frac{dc}{dt} \rangle dt. \]

Put

\[ U := \frac{\frac{dc}{dt}(a)}{(\frac{dc}{dt}(a))}, \quad V := \frac{\frac{dc}{dt}(b)}{(\frac{dc}{dt}(b))}. \]
By $\mathcal{J}$ we denote the directed sectional measure from $U$ to $V$. Then the geodesic curvature $k_g(a)$ of the curve $c$ at $a$ is defined as

$$k_g(a) = \lim_{\delta \alpha \to 0} \frac{\delta \mathcal{J}}{\delta \alpha}.$$ 

Now the Gauss-Bonnet theorem for a domain on an 2-dimensional Lorentzian manifold is stated as follows.

**Theorem 2.1.** (Gauss-Bonnet Theorem). Let $M^2$ be an oriented 2-dimensional Lorentzian manifold and $D$ a simply connected domain on $M^2$ such that the boundary $\partial D$ consists of finite pieces of either spacelike or timelike curves. Then

$$\int_D K \, dS + \int_{\partial D} k_g \, d\alpha + \sum \lambda_i = 2\pi$$

where $\lambda_i$ is the directed sectional measure of the exterior angle at the $i$-th vertex, $K$ the Gaussian curvature and $dS$ the volume element of $M^2$.

3. Curves. Let $c = c(t)$ be a curve in the 3-dimensional Minkowski space $L^3$. If the tangent vector field $dc/dt$ is spacelike, then the curve $c(t)$ is said to be spacelike; similarly for timelike and null.

First we consider spacelike or timelike curve $c(t)$. In this case, we can reparameterize it such that $g(dc/ds, dc/ds) = \varepsilon$ (where $\varepsilon = +1$ if $c$ is spacelike and $\varepsilon = -1$ if $c$ is timelike, respectively). Then this new parameter $s$ is called arclength (or proper time in relativity).

For a timelike curve $c(s)$ with arclength parameter $s$, the Frenet formula is given as

$$\xi_1 := \frac{dc}{ds},$$
$$\frac{d\xi_1}{ds} = k\xi_2,$$
\begin{align}
\frac{d\xi_2}{ds} &= k\xi_1 + w\xi_3, \\
\frac{d\xi_3}{ds} &= -w\xi_2,
\end{align}

where \(\xi_2\) is the unit principal vector field and \(\xi_3\) is the unit binormal vector field, respectively. The scalar function \(k = k(s)\) (resp. \(w = w(s)\)) is called the curvature (resp. torsion) of \(c(s)\).

Next, we consider a null curve \(c(t)\). In this case, we can not have arclength parameter as spacelike or timelike case. However by a special parameter \(s\), we can have the Cartan frame (cf. \([4, 6]\))

\begin{align}
\eta_1 := \frac{dc}{ds}, \\
\frac{d\eta_1}{ds} &= k\xi, \\
\frac{d\eta_2}{ds} &= -w\xi, \\
\frac{d\xi}{ds} &= -w\eta_1 + k\eta_2, \\
g(\eta_i, \eta_i) &= g(\eta_i, \xi) = 0, \quad (i = 1, 2), \\
g(\eta_1, \eta_2) &= -1, \quad g(\xi, \xi) = 1.
\end{align}

The vector field \(\eta_1\) is called null transversal vector field and \(\xi\) is called screen vector field.

4. Jacobi’s Theorem of Timelike Curves. In this section, we shall prove the following theorem.

**Theorem 4.1.** Let \(c(s)\) be a timelike curve with arclength parameter in the 3-dimensional Minkowski space \(\mathbb{L}^3\). Let \(\mathbf{\tau}\) be the image of the Gauss map of the unit principal vector field of \(c(s)\) into the de Sitter space \(S^2_1\). Assume that when the arclength parameter \(s\) varies from \(s = 0\) to \(s = a\), the image \(\mathbf{\tau}\)
is a simple closed curve and not null-homotopic to \( S^1 \) and \( k(0) = k(a) \) and \( w(0) = w(a) \). Then \( \pi \) divides

\[
S^2_1(t_0) = \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 - x_3^2 = 1, |x_3| \leq t_0 \} \subset S^2_1
\]

into two regions with equal areas, where \( t_0 \) is any sufficiently large positive number such that \( \pi \) is contained in the interior of \( S^2_1(t_0) \).

**Proof.** Let \( \bar{s} \) be the arclength parameter of the curve \( \pi \). Since \( c(t) \) satisfies (3.2), we have

\[
\frac{d\bar{s}}{ds} = \frac{d\xi_2}{ds} = (k\xi_1 + w\xi_3) \frac{ds}{d\bar{s}}
\]

and

\[
\frac{d^2\bar{s}}{ds^2} = \left( k\frac{d^2s}{d\bar{s}^2} + k' \left( \frac{ds}{d\bar{s}} \right)^2 \right) \xi_1
\]

\[
+ (k^2 - w^2) \left( \frac{ds}{d\bar{s}} \right)^2 \xi_2 + \left( w\frac{d^2s}{d\bar{s}^2} + w' \left( \frac{ds}{d\bar{s}} \right)^2 \right) \xi_3.
\]

Since the curve \( \pi \) is on the de Sitter space \( S^2_1 \), the geodesic curvature \( k_g(\pi) \) satisfies \( k_g(\pi) = g \left( \frac{d^2\pi}{d\bar{s}^2}, \pi \times \frac{d\pi}{d\bar{s}} \right) \). So, it follows that

\[
k_g(\pi) = kw' \left( \frac{ds}{d\bar{s}} \right)^3 - k'w \left( \frac{ds}{d\bar{s}} \right)^3 = \frac{kw' - k'w}{w^2 - k^2} \left( \frac{ds}{d\bar{s}} \right)
\]

by virtue of (4.1), (4.3) and the equation

\[
\frac{d\bar{s}}{ds} = w^2 - k^2.
\]

By assumption \( w/k > 1 \), we can put \( w = k \cosh \theta, \ -b := \cosh^{-1} \frac{w(0)}{k(0)} \).
b := \cosh^{-1}\frac{w(a)}{k(a)}. Then, we have

(4.3) \int_{\Gamma} k_g(\overline{r}) d\overline{s} = \int_{-b}^{b} \frac{1}{\sinh \theta} d\theta = 0.

Let $S^1(t_0)$ be the circle $x_3 = t_0(> 0)$. Since the geodesic curvature $k_g(S^1(t_0))$ of $S^1(t_0)$ is equal to $-t_0$, we have

(4.4) \int_{S^1(t_0)} k_g(S^1(t_0)) dt = -2\pi t_0.

Let $L$ be a timelike curve on $S^2_1$ and $P$ (resp. $Q$) the crossing point of $L$ to the circle $S^1(t_0)$ (resp. $\overline{c}$). We consider a simply connected domain $D$ constructed by $[S^1(t_0)] + [\overline{PQ}(\subset L)] + [\overline{c}] + [\overline{QP}(\subset L)].$

Applying the Gauss-Bonnet theorem to $D$, we obtain

$$\iint_{D} 1 \cdot dS - 2\pi t_0 + 2\pi = 2\pi,$$

by virtue of (4.3) and (4.4). Therefore

$$[\text{Area} D] = 2\pi t_0 = 2\pi \int_{0}^{\sinh^{-1}t_0} \cosh t dt = \frac{1}{2}[\text{Area} S^2_1(t_0)].$$

This completes the proof.

5. Jacobi’s Theorem of Null Curves. In this section, we shall prove the following Jacobi’s theorem of Cartan framed null curves.

**Theorem 5.1.** Let $c(s)$ be a Cartan framed null curve in the 3-dimensional Minkowski space $\mathbb{L}^3$. Let $\overline{c}$ be the image of the Gauss map of the screen vector field of $c(s)$ into the de Sitter space $S^2_1$. Assume that when the parameter $s$ varies from $s = 0$ to $s = a$, the image $\overline{c}$ is a simple closed curve and not null-homotopic to $S^1$ and $k(0) = k(a)$ and $w(0) = w(a)$. Then $\overline{c}$
divides
\[ S_1^2(t_0) = \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 - x_3^2 = 1, |x_3| \leq t_0\} \subset S_1^2 \]
into two regions with equal areas, where \( t_0 \) is any sufficiently large positive number such that \( \overline{c} \) is contained in the interior of \( S_1^2(t_0) \).

**Proof.** Since \( c(s) \) have Cartan frame, we have
\[ \frac{d\xi}{ds} = (-w \eta_1 + k \eta_2) \frac{ds}{ds} \]
and
\[ \frac{d^2\xi}{ds^2} = \left(-w \frac{d^2s}{ds} - w' \left(\frac{ds}{ds}\right)^2\right) \xi + \left(k \frac{d^2s}{ds} + k' \left(\frac{ds}{ds}\right)^2\right) \xi_2 \]
by virtue of (3.3). Hence the Gaussian curvature \( k_g(\overline{c}) \) satisfies
\[ k_g(\overline{c}) = (\frac{k'}{k} - \frac{w'}{w}) \frac{ds}{ds} \]
so that
\[ \oint g_k(\overline{c}) d\overline{s} = \int \left(\frac{k'}{k} - \frac{w'}{w}\right) ds = 0. \]

Therefore, by a similar calculation, as that of Section 3, we obtain the result.

**References**


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