§ 夾擠(squeeze)原理

(一)

若$b_n \le a_n \le c_n$且$\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} c_n = \alpha$　則$\lim\limits_{n\to\infty} a_n = \alpha$

例

$a_n = \dfrac{1}{\sqrt{4n^2+1}} + \dfrac{1}{\sqrt{4n^2+2}} + \dfrac{1}{\sqrt{4n^2+3}} + \ldots + \dfrac{1}{\sqrt{4n^2+n}}$，求$\lim\limits_{n\to\infty} a_n = ?$

令$b_n = \dfrac{1}{\sqrt{4n^2+n}} + \dfrac{1}{\sqrt{4n^2+n}} + \ldots + \dfrac{1}{\sqrt{4n^2+n}}$，$c_n = \dfrac{1}{\sqrt{4n^2+1}} + \dfrac{1}{\sqrt{4n^2+1}} + \ldots + \dfrac{1}{\sqrt{4n^2+1}}$

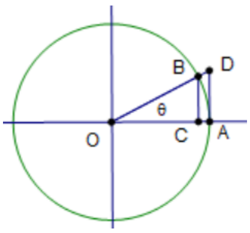
則$b_n \le a_n \le c_n$　且$\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} c_n = \dfrac{1}{2}$

所以$\lim\limits_{n\to\infty} a_n = \dfrac{1}{2}$

(二)

若一區間，$g(x) \le f(x) \le h(x)$　且$\lim\limits_{x\to a} g(x) = \lim\limits_{x\to a} h(x) = \alpha$　則$\lim\limits_{x\to a} f(x) = \alpha$

例　證明$\lim\limits_{\theta\to 0} \dfrac{\sin\theta}{\theta} = 1$



單位圓內，因為$\overline{BC} < AB < \overline{AD}$

(或者說$\Delta OBC <$扇形 OAB$< \Delta OAD$)

$\sin\theta < \theta < \tan\theta$

則$\cos\theta < \dfrac{\sin\theta}{\theta} < 1$

因為$\lim\limits_{\theta\to 0}\cos\theta = 1$　所以$\lim\limits_{\theta\to 0} \dfrac{\sin\theta}{\theta} = 1$

習作

1.　證明$\lim\limits_{x\to 0} x^2 \sin\dfrac{1}{x} = 0$

$\left|\sin\dfrac{1}{x}\right| \le 1$　即$-1 \le \sin\dfrac{1}{x} \le 1$　所以$-x^2 \le x^2 \sin\dfrac{1}{x} \le x^2$

$\lim\limits_{x\to 0}(-x^2) = \lim\limits_{x\to 0} x^2 = 0$

Then　$\lim\limits_{x\to 0} x^2 \sin\dfrac{1}{x} = 0$

2. 兩數列$<a_n>,<b_n>$ 滿足$2b_n+\frac{6n-30}{n}<a_n<4b_n$，$\lim_{n\to\infty}a_n=12$，試證$<b_n>$收斂。

$$\frac{1}{4}a_n<b_n<\frac{1}{2}a_n-(\frac{3n-15}{n})$$

$$\lim_{n\to\infty}\frac{1}{4}a_n=\lim_{n\to\infty}\frac{1}{2}a_n-(\frac{3n-15}{n})=3$$ ，所以$\lim_{n\to\infty}b_n=3$

3. 試證 (1) $3^n\geq n^3$ for $n\geq 3$ (2) $\lim_{n\to\infty}\frac{n^2}{3^n}=0$

4. 利用夾擠原理，求 $\lim_{x\to\infty}\frac{\cos x+5}{x}=$

5. 設$a_n=\frac{\sqrt{1\times 2}+\sqrt{2\times 3}+...+\sqrt{n(n+1)}}{n^2}$ ，求$\lim_{n\to\infty}a_n=$

$$\frac{\sqrt{1\times 1}+\sqrt{2\times 2}+...+\sqrt{n\times n}}{n^2}<a_n<\frac{\sqrt{2\times 2}+\sqrt{3\times 3}+...+\sqrt{(n+1)(n+1)}}{n^2}$$

6. 設$a_n=\frac{1}{n}\sum_{k=1}^{n}\frac{4k}{\sqrt{2n^2+k}}$ ，求$\lim_{n\to\infty}a_n=$

$$\frac{1}{n}\sum_{k=1}^{n}\frac{4k}{\sqrt{2n^2+n}}<a_n<\frac{1}{n}\sum_{k=1}^{n}\frac{4k}{\sqrt{2n^2+1}}$$

7.