

泰勒展開式

$f(x)$ 在 $x=a$ 的泰勒展開式=

The function g has derivatives of all orders, and the Maclaurin series for g is

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+3} = \frac{x}{3} - \frac{x^3}{5} + \frac{x^5}{7} - \dots$$

- (a) Using the ratio test, determine the interval of convergence of the Maclaurin series for g .
- (b) The Maclaurin series for g evaluated at $x = \frac{1}{2}$ is an alternating series whose terms decrease in absolute value to 0. The approximation for $g\left(\frac{1}{2}\right)$ using the first two nonzero terms of this series is $\frac{17}{120}$. Show that this approximation differs from $g\left(\frac{1}{2}\right)$ by less than $\frac{1}{200}$.
- (c) Write the first three nonzero terms and the general term of the Maclaurin series for $g'(x)$.

ANS (a) $-1 \leq x \leq 1$ (b) $\left| g\left(\frac{1}{2}\right) - \frac{17}{120} \right| < \frac{\left(\frac{1}{2}\right)^5}{7} = \frac{1}{224} < \frac{1}{200}$

(c) $g'(x) = \frac{1}{3} - \frac{3}{5}x^2 + \frac{5}{7}x^4 + \dots + (-1)^n \left(\frac{2n+1}{2n+3}\right)x^{2n} + \dots$

2012BC

習作

A function f has derivatives of all orders at $x = 0$. Let $P_n(x)$ denote the n th-degree Taylor polynomial for f about $x = 0$.

(a) It is known that $f(0) = -4$ and that $P_1\left(\frac{1}{2}\right) = -3$. Show that $f'(0) = 2$.

(b) It is known that $f''(0) = -\frac{2}{3}$ and $f'''(0) = \frac{1}{3}$. Find $P_3(x)$.

1. (c) The function h has first derivative given by $h'(x) = f(2x)$. It is known that $h(0) = 7$. Find the third-degree Taylor polynomial for h about $x = 0$.

ANS (a) (b) $P_3(x) = -4 + 2x - \frac{1}{3}x^2 + \frac{1}{18}x^3$ (c) $Q_3(x) = 7 - 4x + 2x^2 - \frac{4}{9}x^3$

2013BC

The Taylor series for a function f about $x = 1$ is given by $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^n}{n} (x-1)^n$ and converges to $f(x)$ for $|x-1| < R$, where R is the radius of convergence of the Taylor series.

- (a) Find the value of R .
- (b) Find the first three nonzero terms and the general term of the Taylor series for f' , the derivative of f , about $x = 1$.
- (c) The Taylor series for f' about $x = 1$, found in part (b), is a geometric series. Find the function f' to which the series converges for $|x-1| < R$. Use this function to determine f for $|x-1| < R$.

2 .
ANS (a) $R = \frac{1}{2}$ (b) $2 - 4(x-1) + 8(x-1)^2, (-1)^{n+1} 2^n (x-1)^{n-1}$ for $n \geq 1$

(c) $f(x) = \ln|2x-1|$ for $|x-1| < \frac{1}{2}$ 2014BC

The Maclaurin series for a function f is given by $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{n} x^n = x - \frac{3}{2}x^2 + 3x^3 - \dots + \frac{(-3)^{n-1}}{n} x^n + \dots$ and converges to $f(x)$ for $|x| < R$, where R is the radius of convergence of the Maclaurin series.

- (a) Use the ratio test to find R .
- (b) Write the first four nonzero terms of the Maclaurin series for f' , the derivative of f . Express f' as a rational function for $|x| < R$.
- (c) Write the first four nonzero terms of the Maclaurin series for e^x . Use the Maclaurin series for e^x to write the third-degree Taylor polynomial for $g(x) = e^x f(x)$ about $x = 0$.

3 .
ANS (a) $R = \frac{1}{3}$ (b) (1) $1 - 3x + 9x^2 - 27x^3$ (2) $f'(x) = \frac{1}{1+3x}$

(c) (1) $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$ (2) $x - \frac{1}{2}x^2 + 2x^3$ 2015BC

The function f has a Taylor series about $x = 1$ that converges to $f(x)$ for all x in the interval of convergence.

It is known that $f(1) = 1$, $f'(1) = -\frac{1}{2}$, and the n th derivative of f at $x = 1$ is given by

$$f^{(n)}(1) = (-1)^n \frac{(n-1)!}{2^n} \text{ for } n \geq 2.$$

- (a) Write the first four nonzero terms and the general term of the Taylor series for f about $x = 1$.
- (b) The Taylor series for f about $x = 1$ has a radius of convergence of 2. Find the interval of convergence. Show the work that leads to your answer.
- (c) The Taylor series for f about $x = 1$ can be used to represent $f(1.2)$ as an alternating series. Use the first three nonzero terms of the alternating series to approximate $f(1.2)$.
- 4 . (d) Show that the approximation found in part (c) is within 0.001 of the exact value of $f(1.2)$.

ANS (a) $f(x) = 1 - \frac{1}{2}(x-1) + \frac{1}{2^2 \cdot 2}(x-1)^2 - \frac{1}{2^3 \cdot 3}(x-1)^3 + \dots + \frac{(-1)^n}{2^n \cdot n}(x-1)^n + \dots$ (b)

$-1 < x \leq 3$ (c) 0.905 (d) $|f(1.2) - T_2(1.2)| \leq \left| \frac{-1}{2^3 \cdot 3} (0.2)^3 \right| = \frac{1}{3000} \leq 0.001$

2016BC

$f(0) = 0, f'(0) = 1, f^{(n+1)}(0) = -nf^{(n)}(0)$ for $n \geq 1$

A function f has derivatives of all orders for $-1 < x < 1$. The derivatives of f satisfy the conditions above. The Maclaurin series for f converges to $f(x)$ for $|x| < 1$.

(a) Show that the first four nonzero terms of the Maclaurin series for f are $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$, and write the general term of the Maclaurin series for f .

(b) Determine whether the Maclaurin series described in part (a) converges absolutely, converges conditionally, or diverges at $x = 1$. Explain your reasoning.

(c) Write the first four nonzero terms and the general term of the Maclaurin series for $g(x) = \int_0^x f(t) dt$.

(d) Let $P_n\left(\frac{1}{2}\right)$ represent the n th-degree Taylor polynomial for g about $x = 0$ evaluated at $x = \frac{1}{2}$, where g is the function defined in part (c). Use the alternating series error bound to show that

5. $\left| P_4\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right) \right| < \frac{1}{500}$.

ANS (a) $\frac{(-1)^{n+1} x^n}{n}$ (b)

For $x = 1$, the Maclaurin series becomes $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$.

The series does not converge absolutely because the harmonic series diverges.

The series alternates with terms that decrease in magnitude to 0, and therefore the series converges conditionally.

(c)

$\frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^5}{20} + \dots + \frac{(-1)^{n+1} x^{n+1}}{(n+1)n} + \dots$

(d) 2017BC

The Maclaurin series for $\ln(1+x)$ is given by

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n+1} \frac{x^n}{n} + \cdots.$$

On its interval of convergence, this series converges to $\ln(1+x)$. Let f be the function defined by

$$f(x) = x \ln\left(1 + \frac{x}{3}\right).$$

- Write the first four nonzero terms and the general term of the Maclaurin series for f .
- Determine the interval of convergence of the Maclaurin series for f . Show the work that leads to your answer.
- Let $P_4(x)$ be the fourth-degree Taylor polynomial for f about $x = 0$. Use the alternating series error bound to find an upper bound for $|P_4(2) - f(2)|$.

6.

ANS (a) (b) $-3 < x \leq 3$ (c) $\frac{8}{81}$ 2018BC