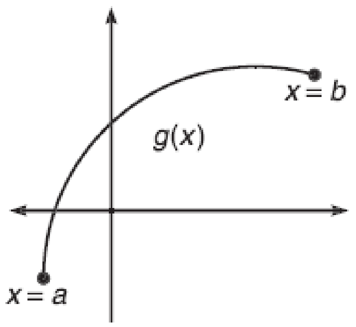
Mean Value theorem

**Mean Value Theorem:** Given a function $g(x)$ that is continuous and differentiable on a closed interval $[a,b]$, there exists at least one $x$ on $[a,b]$ for which _____, the slope of the secant line, equals ______, the slope of the tangent line.

Illustrate the Mean Value Theorem graphically using the graph below of $g$ on $[a,b]$.




**The Mean Value Theorem for Integration:** If $f(x)$ is a continuous function on the interval $[a,b]$, then there exists a real number $c$ on that interval such that $\int_a^b f(x)dx = f(c)(b-a)$.

**The Average Value of a Function:** If $f(x)$ is continuous on $[a,b]$, the average value of the function, $f(c)$, is given by

$$f(c)=\frac{1}{b-a}\cdot\int_a^b f(x)dx$$

**定理 4.2.2.** (平均值定理, Mean Value Theorem) 假設 $y=f(x)$ 滿足以下條件:

(a) 在 $[a,b]$ 上連續,

(b) 在 $(a,b)$ 上可微,

則存在 $c\in(a,b)$, 使得 $f'(c)=\frac{f(b)-f(a)}{b-a}$ [或 $f(b)-f(a)=f'(c)(b-a)$]。

例

If $\int_2^5 f(x)dx = 10$ and $\int_{14}^2 f(x)dx = -29$, what is the average value of $f(x)$ on the interval [5,14]?

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就函數 $f(x) = x^3 - x$ 在 $[0, 2]$ 上驗證平均值定理。

**例 4.2.6.** 假設高速公路限速 90 公里/時, 高雄、台北距離 300 公里。一輛車上午 8:00 從台北出發, 11:00 到達高雄, 則該 車輛必有超速的時刻。

**例 4.2.7.** $f(x) = \frac{x^3}{3} - 3x$ 至少有一水平切線。

**例 4.2.8.** 方程式 $x^3 + 3x - 1 = 0$ 恰有一實根。

| $t$ (minutes) | 0 | 4 | 9 | 15 | 20 |
|---|---|---|---|---|---|
| $W(t)$ (degrees Fahrenheit) | 55.0 | 57.1 | 61.8 | 67.9 | 71.0 |

The temperature of water in a tub at time $t$ is modeled by a strictly increasing, twice-differentiable function $W$, where $W(t)$ is measured in degrees Fahrenheit and $t$ is measured in minutes. At time $t = 0$, the temperature of the water is 55°F. The water is heated for 30 minutes, beginning at time $t = 0$. Values of $W(t)$ at selected times $t$ for the first 20 minutes are given in the table above.

(a) Use the data in the table to estimate $W'(12)$. Show the computations that lead to your answer. Using correct units, interpret the meaning of your answer in the context of this problem.

(c) For $0 \le t \le 20$, the average temperature of the water in the tub is $\frac{1}{20}\int_0^{20} W(t)\,dt$. Use a left Riemann sum with the four subintervals indicated by the data in the table to approximate $\frac{1}{20}\int_0^{20} W(t)\,dt$. Does this approximation overestimate or underestimate the average temperature of the water over these 20 minutes? Explain your reasoning.

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習作

| $t$ (minutes) | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $C(t)$ (ounces) | 0 | 5.3 | 8.8 | 11.2 | 12.8 | 13.8 | 14.5 |

Hot water is dripping through a coffeemaker, filling a large cup with coffee. The amount of coffee in the cup at time $t$, $0 \le t \le 6$, is given by a differentiable function $C$, where $t$ is measured in minutes. Selected values of $C(t)$, measured in ounces, are given in the table above.

(a) Use the data in the table to approximate $C'(3.5)$. Show the computations that lead to your answer, and indicate units of measure.

(b) Is there a time $t$, $2 \le t \le 4$, at which $C'(t) = 2$ ? Justify your answer.

(c) Use a midpoint sum with three subintervals of equal length indicated by the data in the table to approximate the value of $\frac{1}{6}\int_0^6 C(t)\,dt$. Using correct units, explain the meaning of $\frac{1}{6}\int_0^6 C(t)\,dt$ in the context of the problem.

1.

ANS (a)1.6 ounces/min (b) (c)10.1 ounces 2013 AB

Grass clippings are placed in a bin, where they decompose. For $0 \le t \le 30$, the amount of grass clippings remaining in the bin is modeled by $A(t) = 6.687(0.931)^t$, where $A(t)$ is measured in pounds and $t$ is measured in days.

2.

(c) Find the time $t$ for which the amount of grass clippings in the bin is equal to the average amount of grass clippings in the bin over the interval $0 \le t \le 30$.

ANS (c)12.415 2014 AB

| $t$ (hours) | 0 | 1 | 3 | 6 | 8 |
|---|---|---|---|---|---|
| $R(t)$ (liters / hour) | 1340 | 1190 | 950 | 740 | 700 |

Water is pumped into a tank at a rate modeled by $W(t) = 2000e^{-t^2/20}$ liters per hour for $0 \le t \le 8$, where $t$ is measured in hours. Water is removed from the tank at a rate modeled by $R(t)$ liters per hour, where $R$ is differentiable and decreasing on $0 \le t \le 8$. Selected values of $R(t)$ are shown in the table above. At time $t = 0$, there are 50,000 liters of water in the tank.

3. (a) Estimate $R'(2)$. Show the work that leads to your answer. Indicate units of measure.

ANS (a)-120 liters/hr^2 2016AB

| $t$ (years) | 2 | 3 | 5 | 7 | 10 |
|---|---|---|---|---|---|
| $H(t)$ (meters) | 1.5 | 2 | 6 | 11 | 15 |

The height of a tree at time $t$ is given by a twice-differentiable function $H$, where $H(t)$ is measured in meters and $t$ is measured in years. Selected values of $H(t)$ are given in the table above.

(a) Use the data in the table to estimate $H'(6)$. Using correct units, interpret the meaning of $H'(6)$ in the context of the problem.

4. (b) Explain why there must be at least one time $t$, for $2 < t < 10$, such that $H'(t) = 2$.

ANS (a)5/2 2018BC