§ Chapter 6 Techniques of Integration

Differentiation Rules

1. $\frac{d}{dx}(x^n) = nx^{n-1}$
2. $\frac{d}{dx}\left[f(x)g(x)\right] = f(x)g'(x) + g(x)f'(x)$
3. $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{\left[g(x)\right]^2}$
4. $\frac{d}{dx}f\left[g(x)\right] = f'\left[g(x)\right]g'(x)$
5. $\frac{d}{dx}(e^x) = e^x$
6. $\frac{d}{dx}(a^x) = (\ln a)a^x$
7. $\frac{d}{dx}\ln|x| = \frac{1}{x}$
8. $\frac{d}{dx}\log_a x = \frac{1}{\ln a}\cdot\frac{1}{x}$
9. $\frac{d}{dx}(\sin x) = \cos x$
10. $\frac{d}{dx}(\csc x) = -\csc x\cot x$
11. $\frac{d}{dx}(\cos x) = -\sin x$
12. $\frac{d}{dx}(\sec x) = \sec x\tan x$
13. $\frac{d}{dx}(\tan x) = \sec^2 x$
14. $\frac{d}{dx}(\cot x) = -\csc^2 x$
15. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$
16. $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$
17. $\frac{d}{dx}(\csc^{-1} x) = -\frac{1}{x\sqrt{x^2-1}}$
18. $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$
19. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$
20. $\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$

Basic Integration Rules

1. $\int du = u + C$
2. $\int u^n\, du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1$
3. $\int \frac{du}{u} = \ln|u| + C$
4. $\int e^u\, du = e^u + C$
5. $\int a^u\, du = \frac{1}{\ln a}a^u + C$
6. $\int \sin u\, du = -\cos u + C$
7. $\int \cos u\, du = \sin u + C$
8. $\int \sec^2 u\, du = \tan u + C$
9. $\int \csc^2 u\, du = -\cot u + C$
10. $\int \sec u\tan u\, du = \sec u + C$
11. $\int \csc u\cot u\, du = -\csc u + C$
12. $\int \tan u\, du = -\ln|\cos u| + C$
13. $\int \cot u\, du = \ln|\sin u| + C$
14. $\int \sec u\, du = \ln|\sec u + \tan u| + C$
15. $\int \csc u\, du = -\ln|\csc u + \cot u| + C$
16. $\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a} + C$
17. $\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$
18. $\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{u}{a} + C$

Procedures for Fitting Integrands to Basic Rules

Procedure	Example
1. Separating numerator	$\frac{1-2x}{1+x^2}=\frac{1}{1+x^2}-\frac{2x}{1+x^2}$
2. Adding and subtracting terms in numerator	$\frac{1}{1-e^x}=\frac{1-e^x+e^x}{1-e^x}=\frac{1-e^x}{1-e^x}+\frac{e^x}{1-e^x}$
3. Dividing improper fractions	$\frac{x^3-3x}{x^2-1}=x-\frac{2x}{x^2-1}$
4. Completing the square	$\frac{1}{\sqrt{4x-x^2}}=\frac{1}{\sqrt{4-(x-2)^2}}$

Example 1 □ Evaluate $\int \frac{1-2x}{1+x^2}\,dx$.

$$\int \frac{1-2x}{1+x^2}\,dx=\tan^{-1}x-\ln(1+x^2)+C$$

Example 2 □ Evaluate $\int \frac{1}{1-e^x}\,dx$.

$$\int \frac{1}{1-e^x}\,dx=x-\ln(1-e^x)+C$$

Exercise

Evaluate $\int \frac{1}{1+e^x}\,dx$

$$x-\ln(1+e^x)+C$$

Example 3 □ Evaluate $\int \frac{x^3-3x}{x^2-1}\,dx$.

$$\int \frac{x^3-3x}{x^2-1}dx=\frac{1}{2}x^2-\ln(x^2-1)+C$$

Exercise

Evaluate $\int \frac{x^3-x-1}{x^2+1}dx=$

$$\frac{x^2}{2}-\ln(x^2+1)-\arctan x+C$$

Example 4 □ Evaluate $\int \frac{1}{\sqrt{4x-x^2}}\,dx$.

$$\int \frac{1}{\sqrt{4x-x^2}}dx=\sin^{-1}(\frac{x-2}{2})+C$$

Exercise

Evaluate $\int \frac{3}{2x^2+8x+10}dx$

$$\frac{3}{2}\arctan(x+2)+C$$

Exercise

1. $\int \frac{1+\sin x}{\cos^2 x}\,dx =$

(A) $\tan x - \sec x \tan x + C$

(B) $\tan x + \sec x + C$

(C) $\tan x + \sec^2 x + C$

(D) $\ln(1+\cos^2 x) + C$

2. $\int \frac{e^{2x}}{1+e^x}\,dx =$

(A) $e^{2x} + \ln(1+e^x) + C$

(B) $e^{2x} - \ln(1+e^x) + C$

(C) $2e^{2x} - \ln(1+e^x) + C$

(D) $e^x - \ln(1+e^x) + C$

3. $\int 2\tan x \ \ln(\cos x)\,dx =$

(A) $\cos x \ [\ln(\cos x)] + C$

(B) $\sin x [\ln(\cos x)] + C$

(C) $-[\ln(\cos x)]^2 + C$

(D) $[\ln(\sin x)]^2 + C$

4. $\int_2^3 \frac{1}{x^2 - 4x + 5}\,dx =$

(A) $\frac{\pi}{4}$ (B) $1 - \frac{\pi}{4}$ (C) $1 + \frac{\pi}{6}$ (D) $1 + \frac{\pi}{4}$

5. $\int \frac{2x}{x^2+2x+1}\,dx =$

(A) $-\operatorname{arc\,cot} x - \frac{1}{x+1} + C$

(B) $\arctan x + \frac{1}{x+1} + C$

(C) $2\ln|x+1| - \frac{2}{(x+1)^2} + C$

(D) $2\ln|x+1| + \frac{2}{x+1} + C$

6. The region bounded by $y = \frac{\sin x}{\sqrt{\cos x}}$, $x = 0$, $x = \frac{\pi}{4}$, and the x-axis is revolved around the x-axis. What is the volume of the resulting solid?

§ 6.2 Trigonometric Integration 暫略

§ 6.3 Trigonometric Substitution

Example 1 □ Evaluate $\int_0^2 \sqrt{4-x^2}\,dx$.

$2\sin^{-1}\frac{x}{2} + \frac{x\sqrt{4-x^2}}{2} + C$ ； π

Example 2 □ Evaluate $\int \frac{dx}{\sqrt{9+x^2}}$.

$\ln\left|\frac{\sqrt{9+x^2}}{3} + \frac{x}{3}\right| + C$

Example ：Evaluate $\int \frac{x-2}{\sqrt{9-x^2}}dx =$

$-(\sqrt{9-x^2}+2\sin^{-1}\frac{x}{3})+C$