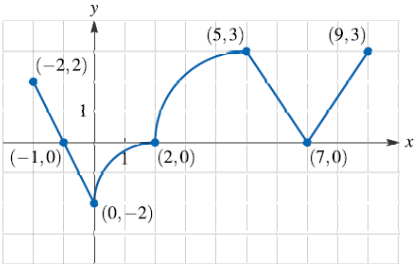
AP Calculus Mock Exam

AB 1

The continuous function f has domain $-2 \leq x \leq 9$. The graph of f, consisting of three line segments and two quarter circles, is shown in the figure.



Graph of f

Let g be the function defined by $g(x) = \int_0^x f(t)\,dt$ for $-2 \leq x \leq 9$.

(a) Find the x-coordinate of each critical point of g on the interval $-2 \leq x \leq 9$.

(b) Classify each critical point from part (a) as the location of a relative minimum, a relative maximum, or neither for g. Justify your answers.

(c) For $-2 \leq x \leq 9$, on what open intervals is g increasing and concave down? Give a reason for your answer.

(d) Find the value of $g(-1)$. Show the computations that lead to your answer.

(e) Find the value of $g(2)$. Show the computations that lead to your answer.

(f) Find the absolute maximum value of g over the interval $-2 \leq x \leq 5$.

(g) Find the value of $g''(6)$, or explain why it does not exist.

(h) Must there exist a value of d, for $0 < d < 2$, such that $g'(d)$ is equal to the average rate of change of g over the interval $0 \leq x \leq 2$? Justify your answer.

(i) Find $\lim_{x \to 0} \frac{3x + g(x)}{\sin x}$. Show the computations that lead to your answer.

(j) The function h is defined by $h(x) = x \cdot g(x^2)$. Find $h'(\sqrt{2})$. Show the computations that lead to your answer.

(a) x=-1,2,7

(b) At $x = -1$, g has a relative maximum because $g'(x) = f(x)$ changes from positive to negative there.

At $x = 2$, g has a relative minimum because $g'(x) = f(x)$ changes from negative to positive there.

At $x = 7$, g has neither because $g'(x) = f(x)$ does not change sign there.

(c) g is increasing where $g' = f$ is positive.

g is concave down where $g' = f$ is decreasing.

g is increasing and concave down on the intervals $(-2, -1)$ and $(5, 7)$.

(d)1

(e) $\pi - 4$

(f) The absolute maximum value occurs at an endpoint of the interval or a critical point.

Consider a table of values.

x	$g(x)$
-2	0
-1	1
2	$\pi - 4$
5	$\pi - 4 + \frac{1}{4}\pi 3^2 = \frac{13}{9}\pi - 4$

The absolute maximum value of g is $\frac{13}{9}\pi - 4$.

(g) $-\frac{3}{2}$

(h) $g' = f \quad \Rightarrow \quad g$ is differentiable on $0 < x < 2 \quad \Rightarrow \quad g$ is continuous on $0 \leq x \leq 2$

Therefore, the Mean Value Theorem can be applied to g on the interval $0 \leq x \leq 2$ to guarantee that there exists a value of d, for $0 < d < 2$, such that $g'(d)$ equals the average rate of change of g over the interval $0 \leq x \leq 2$.

(i) $\lim_{x \to 0} (3x + g(x)) = 0$

$\lim_{x \to 0} \sin x = 0$

Therefore the limit $\lim_{x \to 0} \frac{3x + g(x)}{\sin x}$ is in the indeterminate form $\frac{0}{0}$ and L'Hospital's Rule can be applied.

$$\lim_{x \to 0} \frac{3x + g(x)}{\sin x} = \lim_{x \to 0} \frac{3 + g'(x)}{\cos x} = \frac{3 + g'(0)}{\cos 0}$$
$$= \frac{3 + f(0)}{\cos 0} = \frac{3 + -2}{1} = 1$$

(j) $h'(x) = 1 \cdot g(x^2) + x \cdot g'(x^2) \cdot 2x$
$$= g(x^2) + 2x^2 f(x^2)$$

$$h'(\sqrt{2}) = g(2) + 2 \cdot 2 \cdot f(2)$$
$$= (\pi - 4) + 4 \cdot 0 = \pi - 4$$

AB 2

t	0	2	6	8	10	12
$y'(t)$	4	8	-2	3	-1	-5

The vertical position of a particle moving along the y-axis is modeled by a twice-differentiable function $y(t)$ where t is measured in seconds and $y(t)$ is measured in meters. Selected values of $y'(t)$, the derivative of $y(t)$, over the interval $0 \leq t \leq 12$ seconds are shown in the table above. The position of the particle at time $t = 12$ is $y(12) = -3$.

(a) Use a locally linear approximation of y at $t = 12$ to approximate $y(11.8)$.

(b) Approximate $y''(4)$ using the average rate of change of $y'(t)$ on the interval $2 \leq t \leq 6$.

(c) Using correct units, explain the meaning of $y''(4)$ in the context of the problem.

(d) Find the average value of the acceleration of the particle over the interval $[0, 12]$.

(e) Using a midpoint Riemann sum and three subintervals of equal length, approximate $\int_0^{12} y'(t)\,dt$.

(f) Using correct units, explain the meaning of $\int_0^{12} y'(t)\,dt$ in the context of the problem.

(g) Explain why there must be at least three times t in the interval $0 < t < 12$ such that $y'(t) = 0$.

(h) Explain why there must be at least two times t in the interval $0 < t < 12$ such that $y''(t) = 0$.

(a) -2

(b) $y''(4) \approx \dfrac{y'(6) - y'(2)}{6 - 2} = \dfrac{-2 - 8}{4} = -2.5$

(b)

(c) $y''(4)$ is the acceleration of the particle in meters per second per second at time $t = 4$ seconds.

(c)

(d)
$$\frac{1}{12}\int_0^{12} y''(t)\,dt = \frac{1}{12}\Big[y'(t)\Big]_0^{12}$$
$$= \frac{1}{12}[y'(12) - y'(0)]$$
$$= \frac{1}{12}[-5 - 4] = -\frac{3}{4}$$

(d)

(e) $\displaystyle\int_0^{12} y'(t)\,dt$

$$\approx y'(2)\cdot(4-0) + y'(6)\cdot(8-4) + y'(10)\cdot(12-8)$$
$$= (8)(4) + (-2)(4) + (-1)(4) = 20$$

(e)

(f) $\displaystyle\int_0^{12} y'(t)\,dt$ is the vertical displacement (change in position), in meters, of the particle over the time interval $0 \le t \le 12$ seconds.

(g) **(g)** y is twice differentiable $\Rightarrow y'$ is differentiable
$\Rightarrow y'$ is continuous over the interval $[0, 12]$.

$y'(t)$ changes from positive to negative on the interval $[2, 6]$.

$y'(t)$ changes from negative to positive on the interval $[6, 8]$.

$y'(t)$ changes from positive to negative on the interval $[8, 10]$.

By the Intermediate Value Theorem, there must be times a, b, and c, such that $2 < a < 6 < b < 8 < c < 10$, and $y'(a) = y'(b) = y'(c) = 0$

(h) **(h)** y is twice differentiable $\Rightarrow y'$ is differentiable
$\Rightarrow y'$ is continuous over the interval $[0, 12]$.

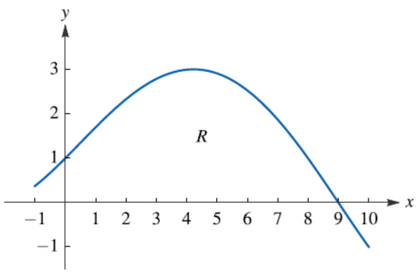
From part (g): there are times $t = a, t = b$, and $t = c$ such that $0 < a < b < c < 12$ and $y'(a) = y'(b) = y'(c) = 0$.

$$\frac{y'(b) - y'(a)}{b - a} = 0 \quad \text{and} \quad \frac{y'(c) - y'(b)}{c - b} = 0$$

By the Mean Value Theorem (or Rolle's Theorem) there must be a time $t = t_1$ such that $a < t_1 < b$ and $y''(t_1) = 0$ and a time $t = t_2$ such that $b < t_2 < c$ and $y''(t_2) = 0$

BC 1

The graph of g', the derivative of the twice-differentiable function g, is shown for $-1 < x < 10$. The graph of g' has exactly one horizontal tangent line, at $x = 4.2$.



Graph of g'

Let R be the region in the first quadrant bounded by the graph of g' and the x-axis from $x = 0$ to $x = 9$. It is known that $g(0) = -7$, $g(9) = 12$, and $\int_0^9 g(x)\,dx = 27.6$.

(a) Find all values of x in the interval $-1 < x < 10$, if any, at which g has a critical point. Classify each critical point as the location of a relative minimum, relative maximum, or neither, Justify your answers.

(b) How many points of inflection does the graph of g have on the interval $-1 < x < 10$? Give a reason for your answer.

(c) Find the area of the region R.

(d) Write an expression that represents the perimeter of the region R. Do not evaluate this expression.

(e) Must there exist a value of c, for $0 < c < 9$, such that $g(c) = 0$? Justify your answer.

(f) Evaluate $\int_0^9 \left[\frac{1}{2}g(x) - \sqrt{x}\right] dx$. Show the computations that lead to your answer.

(g) Evaluate $\lim_{x\to 0} \frac{x\cos x}{g(x) + 2x - 1}$. Show the computations that lead to your answer.

(h) Let h be the function defined by $h(x) = \int_{x^2}^{0} g(t)\,dt$. Find $h'(3)$. Show the computations that lead to your answer.

(i) The region R is the base of a solid. For this solid, at each x the cross section perpendicular to the x-axis is a right triangle with height x and base in the region R. The volume of the solid is given by $\int_0^9 A(x)\,dx$. Write an expression for $A(x)$.

(j) Find the volume of the solid described in part (h). Show the computations that lead to your answer.

(k) Find the value of $\int_0^9 \frac{g''(x)}{g'(x)}\,dx$ or show that it does not exist.

(l) If $g''(0) = 0.7$, find the second degree Taylor polynomial for g about $x = 0$.

(a) $g'(x) = 0$: $x = 9$

$g'(x)$ DNE: none

g has a critical point at $x = 9$.

At $x = 9$, g has a relative maximum because $g'(x)$ changes from positive to negative there.

(b) The graph of g has a point of inflection where g' changes from increasing to decreasing or from decreasing to increasing.

g' changes from increasing to decreasing at $x = 4.2$.

Therefore the graph of g has one point of inflection at the point where $x = 4.2$.

(c) 19

(d) $P = 1 + 9 + \int_0^9 \sqrt{1 + g''(x)^2}\, dx$

(e) Since g is differentiable, then g is continuous on $0 \le x \le 9$.

$g(0) = -7 < 0 < 12 = g(9)$

By the Intermediate Value Theorem, there exists a value of c, for $0 < c < 9$, such that $g(c) = 0$.

(f) -4.2

(g) **(g)** $\lim_{x\to 0} (x \cos x) = 0$

$\lim_{x\to 0} (g(x) + 2x - 1) = 0$

Therefore the limit $\lim_{x\to 0} \frac{x \cos x}{g(x) + 2x - 1}$ is in the indeterminate form $\frac{0}{0}$ and L'Hospital's Rule can be applied.

$$\lim_{x\to 0} \frac{x \cos x}{g(x) + 2x - 1} = \lim_{x\to 0} \frac{x \cdot (-\sin x) + 1 \cdot \cos x}{g'(x) + 2}$$

$$= \frac{0 \cdot (-\sin 0) + 1 \cdot \cos 0}{g'(0) + 2} = \frac{1}{3}$$

(h) $h'(x) = \frac{d}{dx}\left[\int_{x^2}^{0} g(t)\,dx\right]$

$$= -\frac{d}{dx}\left[\int_{0}^{x^2} g(t)\,dt\right]$$

$$= -g(x^2)\cdot(2x) = -2xg(x^2)$$

(h) $h'(3) = -2\cdot 3\cdot g(9) = -6\cdot 12 = -72$

(i) $A(x)$ represents the area of a right triangle at each x.

$$A(x) = \frac{1}{2}xg'(x)$$

(j) $V = \int_0^9 A(x)\,dx = \frac{1}{2}\int_0^9 xg'(x)\,dx$

Use integration by parts.

$u = x \qquad dv = g'(x)\,dx$

$du = dx \qquad v = \int g'(x)\,dx = g(x)$

$$V = \frac{1}{2}\left(\left[x\cdot g(x)\right]_0^9 - \int_0^9 g(x)\,dx\right)$$

$$= \frac{1}{2}\left([9\cdot g(9) - 0\cdot g(0)] - 27.6\right)$$

(j) $= \frac{1}{2}(9\cdot 12 - 27.6) = 40.2$

(k) $\int_0^9 \frac{g''(x)}{g'(x)}\,dx = \lim_{t\to 9^-}\int_0^t \frac{g''(x)}{g'(x)}\,dx$

Let $u = g'(x)$, then $du = g''(x)\,dx$ and $dx = \frac{du}{g''(x)}$

$$\int \frac{g''(x)}{g'(x)}\,dx = \int \frac{g''(x)}{u}\cdot\frac{du}{g''(x)} = \int \frac{du}{u}$$

$$= \ln|u| = \ln|g'(x)|$$

$$\lim_{t\to 9^-}\int_0^9 \frac{g''(x)}{g'(x)}\,dx = \lim_{t\to 9^-}\left[\ln g'(x)\right]_0^t$$

$$= \lim_{t\to 9^-}[\ln g'(t) - \ln g'(0)]$$

$$= \lim_{t\to 9^-}\ln g'(t) = -\infty$$

(k) Therefore the improper integral does not exist.

(l) $g(0) = -7, \quad g'(0) = 1, \quad g''(0) = 0.7$

$$\begin{aligned} T_2(x) &= g(0) + g'(0)x + \frac{g''(0)}{2!}x^2 \\ &= -7 + 1 \cdot x + \frac{0.7}{2}x^2 \\ &= -7 + x + 0.35x^2 \end{aligned}$$

BC 2

t	0	2	6	8	10	12
$y'(t)$	4	8	-2	3	-1	-5

A particle moves in the coordinate plane with position $(x(t), y(t))$ at time t, where t is measured in seconds and $x(t)$ and $y(t)$ are twice-differentiable functions, both measured in meters.

For all times t, the x-coordinate of the particle's position has derivative $x'(t) = \dfrac{t}{\sqrt{t^2 + 25}}$. Selected values of $y'(t)$, the derivative of $y(t)$, over the interval $0 \le t \le 12$ seconds are shown in the table. The position of the particle at time $t = 12$ is $(x(12), y(12)) = (4, -3)$.

(a) Using correct units, find the speed of the particle at time $t = 6$.

(b) Find the exact value of $x(4)$, the x-coordinate of the position of the particle at time $t = 4$.

(c) Using Euler's method, starting at time $t = 12$ with 4 steps of equal size, approximate $y(4)$, the y-coordinate of the position of the particle at $t = 4$.

(d) Find an equation of the line tangent to the path of the particle at time $t = 12$.

(e) Let $r(t)$ be the distance between the particle and the origin $(0, 0)$ at time t. Find $r'(12)$.

(f) Is the particle moving closer or further from the origin at time $t = 12$? Justify your answer.

(g) Given $y''(12) = -2$ and $y'''(12) = 8$, find the third-degree Taylor polynomial approximation for y about $t = 12$.

(h) Suppose that over the time interval $[12, 15]$ the y-coordinate of the position of the particle is the same as the Taylor polynomial approximation found in part (g). Set up but do not evaluate an expression that represents the total distance traveled by the particle over the interval $[12, 15]$.

(a) $x'(6) = \frac{6}{\sqrt{6^2+25}} = \frac{6}{\sqrt{61}}$ and $y'(6) = -2$

speed $= \sqrt{[x'(6)]^2 + [y'(6)]^2}$

$= \sqrt{\left(\frac{6}{\sqrt{61}}\right)^2 + (-2)^2}$

$= \sqrt{\frac{36}{61} + 4} = 2\sqrt{\frac{70}{61}}$ m/s

2: { 1 : expression for speed
1 : answer with units }

(a)

(b) $x(4) = \sqrt{41} - 9$

(c) $y(10) \approx y(12) + (-2) \cdot y'(12) = -3 + (-2)(-5) = 7$

$y(8) \approx y(10) + (-2) \cdot y'(10) = 7 + (-2)(-1) = 9$

$y(6) \approx y(8) + (-2) \cdot y'(8) = 9 + (-2)(3) = 3$

$y(4) \approx y(6) + (-2)y'(6) = 3 + (-2)(-2) = 7$

2: { 1 : First step in Euler's method
1 : answer }

(c)

(d) $\left.\frac{dy}{dx}\right|_{t=12} = \frac{y'(12)}{x'(12)} = \frac{-5}{\frac{12}{\sqrt{12^2+25}}} = -\frac{65}{12}$

An equation of the tangent line at the position $(x(12), y(12)) = (4, -3)$ is

$y = -\frac{65}{12}(x-4) - 3$

2: { 1 : slope
1 : tangent line equation }

(d)

(e) $r(t) = \sqrt{x(t)^2 + y(t)^2}$

$r'(t) = \frac{1}{2}(x(t)^2 + y(t)^2)^{-1/2}(2x(t)x'(t) + 2y(t)y'(t))$

$= \frac{x(t)x'(t) + y(t)y'(t)}{\sqrt{x(t)^2 + y(t)^2}}$

$r'(12) = \frac{x(12)x'(12) + y(12)y'(12)}{\sqrt{x(12)^2 + y(12)^2}}$

$= \frac{(4)\left(\frac{12}{13}\right) + (-3)(-5)}{\sqrt{4^2 + (-3)^2}}$

$= \frac{243}{65}$

(e)

(f) $r'(12) = \frac{243}{65} > 0$

The distance r to the origin is increasing at time $t = 12$.

Therefore the particle is moving away from the origin at time $t = 12$ seconds.

(g) $= -3 - 5(t-12) - (t-12)^2 + \frac{4}{3}(t-12)^3$

(h) For $12 \le t \le 15$,

$$y(t) = g(t) = -3 - 5(t-12) - (t-12)^2 + \tfrac{4}{3}(t-12)^3$$

$$x'(t) = \frac{t}{\sqrt{t^2+25}}$$

$$y'(t) = g'(t) = -5 - 2(t-12) + 4(t-12)^2$$

The distance traveled by the particle over the time interval $[12, 15]$ is

$$\int_{12}^{15} \sqrt{\left[\frac{t}{\sqrt{t^2+25}}\right]^2 + \left[-5 - 2(t-12) + 4(t-12)^2\right]^2}\, dt$$