Question 1

Grass clippings are placed in a bin, where they decompose. For $0 \le t \le 30$, the amount of grass clippings remaining in the bin is modeled by $A(t) = 6.687(0.931)^t$, where $A(t)$ is measured in pounds and t is measured in days.

(a) Find the average rate of change of $A(t)$ over the interval $0 \le t \le 30$. Indicate units of measure.

(b) Find the value of $A'(15)$. Using correct units, interpret the meaning of the value in the context of the problem.

(c) Find the time t for which the amount of grass clippings in the bin is equal to the average amount of grass clippings in the bin over the interval $0 \le t \le 30$.

(d) For $t > 30$, $L(t)$, the linear approximation to A at $t = 30$, is a better model for the amount of grass clippings remaining in the bin. Use $L(t)$ to predict the time at which there will be 0.5 pound of grass clippings remaining in the bin. Show the work that leads to your answer.

Grass clippings 草屑　bin 垃圾桶　decompose 分解

(a)均值定理 $\frac{A(30)-A(0)}{30-0}\approx$-0.197

(b)導數的意義

$A'(15)=$-0.164，在第 15 天時垃圾桶中草的量以每天 0.164 磅的速率減少

(c)均值定理

$A(t)=\frac{1}{30}\int_0^{30} A(t)dt$ ，t=12.415

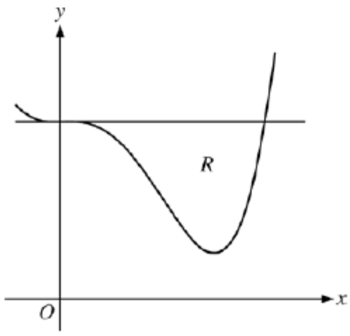
(d) $L(t)=A(30)+A'(30)(t-30)$

$A'(30)=-0.0055976, A(30)=0.782928$

L(t)=0.5，t=35.054

Question 2

Let R be the region enclosed by the graph of $f(x) = x^4 - 2.3x^3 + 4$ and the horizontal line $y = 4$, as shown in the figure above.



(a) Find the volume of the solid generated when R is rotated about the horizontal line $y = -2$.

(b) Region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is an isosceles right triangle with a leg in R. Find the volume of the solid.

(c) The vertical line $x = k$ divides R into two regions with equal areas. Write, but do not solve, an equation involving integral expressions whose solution gives the value k.

Isosceles 等腰三角形 cross section 橫截面

(a)旋轉體體積

f(x)=4，x=0 and 2.3

旋轉體體積$=\pi\int_0^{2.3}[(4+2)^2-(f(x)+2)^2]dx=98.868$

(b)體積

$$\int_0^{2.3}\frac{1}{2}(4-f(x))^2dx=3.574$$

(c)積分

$$\int_0^{k}(4-f(x))dx=\int_k^{2.3}(4-f(x))dx$$

Question 3

The function f is defined on the closed interval $[-5, 4]$. The graph of f consists of three line segments and is shown in the figure above.

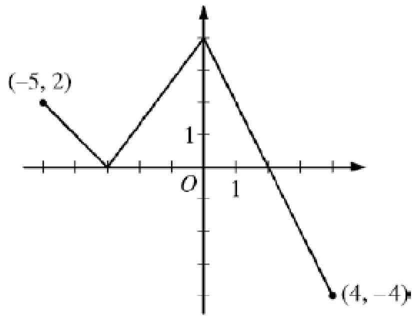
Let g be the function defined by $g(x) = \int_{-3}^{x} f(t)\, dt$.

(a) Find $g(3)$.

(b) On what open intervals contained in $-5 < x < 4$ is the graph of g both increasing and concave down? Give a reason for your answer.

(c) The function h is defined by $h(x) = \frac{g(x)}{5x}$. Find $h'(3)$.

(d) The function p is defined by $p(x) = f\left(x^2 - x\right)$. Find the slope of the line tangent to the graph of p at the point where $x = -1$.



Graph of f

(a) 面積=6+4-1=9

(b) 函數的性質

函數漸增且凹向下 -5<x<-3 and 0<x<2

(c) 微分的公式 $-\frac{1}{3}$

(d) Chain rule

$p'(x) = f'(x^2 - x)(2x-1)$

切線斜率 $p'(-1) = f'(2)\times(-3) = 6$

Question 4

Train A runs back and forth on an east-west section of railroad track. Train A's velocity, measured in meters per minute, is given by a differentiable function $v_A(t)$, where time t is measured in minutes. Selected values for $v_A(t)$ are given in the table above.

t (minutes)	0	2	5	8	12
$v_A(t)$ (meters/minute)	0	100	40	−120	−150

(a) Find the average acceleration of train A over the interval $2 \le t \le 8$.

(b) Do the data in the table support the conclusion that train A's velocity is -100 meters per minute at some time t with $5 < t < 8$? Give a reason for your answer.

(c) At time $t = 2$, train A's position is 300 meters east of the Origin Station, and the train is moving to the east. Write an expression involving an integral that gives the position of train A, in meters from the Origin Station, at time $t = 12$. Use a trapezoidal sum with three subintervals indicated by the table to approximate the position of the train at time $t = 12$.

(d) A second train, train B, travels north from the Origin Station. At time t the velocity of train B is given by $v_B(t) = -5t^2 + 60t + 25$, and at time $t = 2$ the train is 400 meters north of the station. Find the rate, in meters per minute, at which the distance between train A and train B is changing at time $t = 2$.

Trapezoidal rule 梯形法

運動方程式

(a) 平均加速度 $= \dfrac{v(8)-v(2)}{8-2} = -\dfrac{110}{3}$

(b) 中間值定理(Intermediate value theorem)

(c) 位置函數與速度函數 積分 原點西方 150m

$$s(12) = s(2) + \int_2^{12} v(t)dt$$

$$\int_2^{12} v(t)dt = \ldots = -450$$

(d) 兩質點距離的變化率 160m/min

A 在 x,B 在 y 則兩者的距離 z，$z^2 = x^2 + y^2$

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

x=300，y=400 則 z=500

$$v_A(2) = 100, v_B(2) = 125$$

$$\left.\frac{dz}{dt}\right|_{t=2} = 160m/\min$$

Question 5

x	-2	$-2 < x < -1$	-1	$-1 < x < 1$	1	$1 < x < 3$	3
$f(x)$	12	Positive	8	Positive	2	Positive	7
$f'(x)$	-5	Negative	0	Negative	0	Positive	$\frac{1}{2}$
$g(x)$	-1	Negative	0	Positive	3	Positive	1
$g'(x)$	2	Positive	$\frac{3}{2}$	Positive	0	Negative	-2

The twice-differentiable functions f and g are defined for all real numbers x. Values of f, f', g, and g' for various values of x are given in the table above.

(a) Find the x-coordinate of each relative minimum of f on the interval $[-2, 3]$. Justify your answers.

(b) Explain why there must be a value c, for $-1 < c < 1$, such that $f''(c) = 0$.

(c) The function h is defined by $h(x) = \ln(f(x))$. Find $h'(3)$. Show the computations that lead to your answer.

(d) Evaluate $\int_{-2}^{3} f'(g(x))\,g'(x)\,dx$.

(a) f(x)在 x=1 有相對極小值

(b) mean value theorem

(c) chain rule $\frac{1}{14}$

(d) 反導函數 -6

Question 6

Consider the differential equation $\frac{dy}{dx} = (3 - y)\cos x$. Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(0) = 1$. The function f is defined for all real numbers.

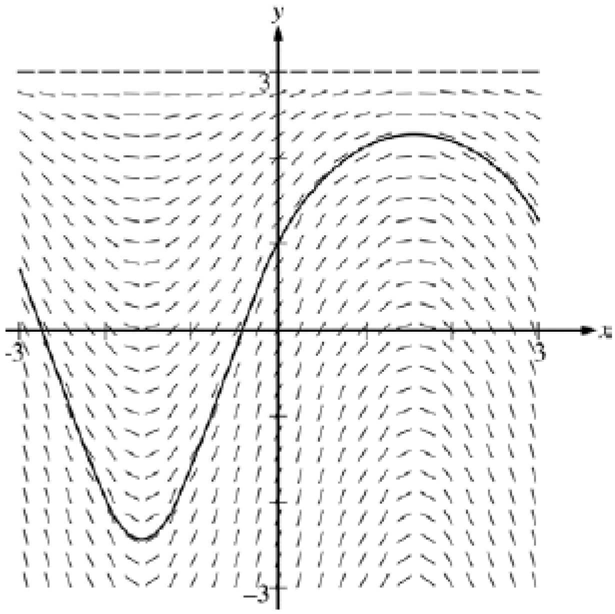
(a) A portion of the slope field of the differential equation is given below. Sketch the solution curve through the point $(0, 1)$.

(b) Write an equation for the line tangent to the solution curve in part (a) at the point $(0, 1)$. Use the equation to approximate $f(0.2)$.

(c) Find $y = f(x)$, the particular solution to the differential equation with the initial condition $f(0) = 1$.

微分方程 slope field

(a)通過(0,1)的解曲線



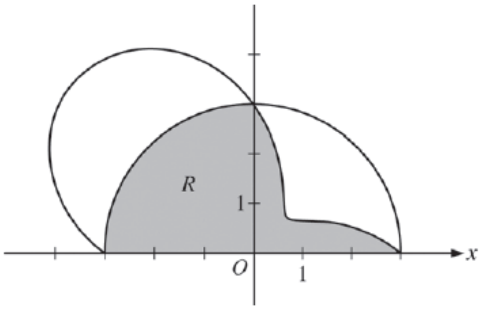
(b)求切線方程式與近似值

切線 y=2x+1

$f(0.2) \approx 1.4$

(c)分離變數法解微分方程 $y = 3 - 2e^{-\sin x}$

BC



2. The graphs of the polar curves $r = 3$ and $r = 3 - 2\sin(2\theta)$ are shown in the figure above for $0 \le \theta \le \pi$.

(a) Let R be the shaded region that is inside the graph of $r = 3$ and inside the graph of $r = 3 - 2\sin(2\theta)$. Find the area of R.

(b) For the curve $r = 3 - 2\sin(2\theta)$, find the value of $\frac{dx}{d\theta}$ at $\theta = \frac{\pi}{6}$.

(c) The distance between the two curves changes for $0 < \theta < \frac{\pi}{2}$. Find the rate at which the distance between the two curves is changing with respect to θ when $\theta = \frac{\pi}{3}$.

(d) A particle is moving along the curve $r = 3 - 2\sin(2\theta)$ so that $\frac{d\theta}{dt} = 3$ for all times $t \ge 0$. Find the value of $\frac{dr}{dt}$ at $\theta = \frac{\pi}{6}$.

(a)極座標求面積

$$\frac{9\pi}{4} + \frac{1}{2}\int_0^{\frac{\pi}{2}} [3 - 2\sin(2\theta)]^2 d\theta = 9.708$$

(b)微分

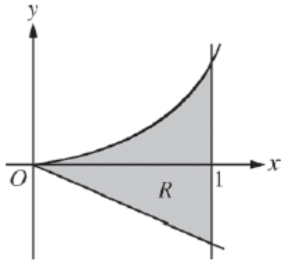
$$x = r\cos\theta = (3 - 2\sin(2\theta))\cos\theta$$

$$\frac{dx}{d\theta}_{\theta=\frac{\pi}{6}} = -2.366$$

(c) $D = 3 - [3 - 2\sin(2\theta)] = 2\sin(2\theta)$

$$\left.\frac{dD}{d\theta}\right|_{\theta=\frac{\pi}{3}} = -2$$

(d) Chain rule $\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt}$ -6



5. Let R be the shaded region bounded by the graph of $y = xe^{x^2}$, the line $y = -2x$, and the vertical line $x = 1$, as shown in the figure above.

(a) Find the area of R.

(b) Write, but do not evaluate, an integral expression that gives the volume of the solid generated when R is rotated about the horizontal line $y = -2$.

(c) Write, but do not evaluate, an expression involving one or more integrals that gives the perimeter of R.

(a)積分求面積

$$\int_0^1 [xe^{x^2} - (-2x)]dx = \frac{e+1}{2}$$

(b)旋轉體體積

(c) 弧長

$$s = \int \sqrt{1 + (f'(t))^2}\,dt = \int \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\,dt$$

$$\text{Perimeter} = \sqrt{5} + 2 + e + \int_0^1 \sqrt{1 + \left[e^{x^2}\left(1 + 2x^2\right)\right]^2}\,dx$$

The Taylor series for a function f about $x = 1$ is given by $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^n}{n}(x-1)^n$ and converges to $f(x)$ for $|x-1| < R$, where R is the radius of convergence of the Taylor series.

(a) Find the value of R.

(b) Find the first three nonzero terms and the general term of the Taylor series for f', the derivative of f, about $x = 1$.

(c) The Taylor series for f' about $x = 1$, found in part (b), is a geometric series. Find the function f' to which the series converges for $|x-1| < R$. Use this function to determine f for $|x-1| < R$.

泰勒展開式

(a) Ratio test 求收斂半徑

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|<1 \qquad R=\frac{1}{2}$$

(b) The first three nonzero terms are

$2-4(x-1)+8(x-1)^2$.

The general term is $(-1)^{n+1}2^n(x-1)^{n-1}$ for $n \geq 1$.

(c) 無窮等比級數 $f(x)=\ln|2x-1|$ for $|x-1|<\frac{1}{2}$