§ Pauli Spinors

4.1

1. Prove that the Pauli matrices σ_j ，j=1，2，3，satisfy the relation

$$\sigma_j\sigma_k = i\sum_{m=1}^{3}\varepsilon_{jkm}\sigma_m + \delta_{jk}I$$

已知$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\sigma_1\sigma_2 = i\sigma_3, \sigma_2\sigma_3 = i\sigma_1, \sigma_1\sigma_3 = i\sigma_2$ ， $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = I$

2. Prove that $\sigma_1\sigma_2\sigma_3 = iI$

3. Prove that every matrix M of order 2 can be put into the form $M = a_0 I + a \bullet \sigma$ where $a = (a_1, a_2, a_3), \sigma = (\sigma_1, \sigma_2, \sigma_3)$

For $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ ，then $a_0 = \frac{1}{2}(a+d), a_3 = \frac{1}{2}(a-d), a_1 = \frac{1}{2}(b+c), a_2 = \frac{i}{2}(b-c)$

4.2 Let us write $|+\rangle = (1,0), |-\rangle = (0,1)$ for the orthonormal spinors which can serve as a basis of the space of spinors。

1. Show that $|+\rangle$ and $|-\rangle$ are eigenvectors of the Pauli matrix σ_z and calculate its eigenvectors。

$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} = -\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ 所以對應的 eigenvalues 是 1，-1

2. Calculate the eigenvalues and the normed eigenspinors of σ_x and σ_y in the basis $\{|+\rangle, |-\rangle\}$。

對σ_x而言，$\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0$ ，$\lambda^2 = 1$

$\lambda = 1$ ，$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}$ ，normed eigenvector is $\frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$

$\lambda = -1$ ，normed eigenvector is $\frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)$

4.3 Consider a unitary vector u of which the components in spherical coordinates

θ,φ are $u_x=\sin\theta\cos\varphi, u_y=\sin\theta\sin\varphi, u_z=\cos\theta$

Let S_x, S_y, S_z be the operators the matrices of which are the Pauli matrices in the

basis $\{|+\rangle, |-\rangle\}$

1. Write S_u for the operator $S_u = u\bullet S = u_x S_x + u_y S_y + u_z S_z$。Calculate the

   matrix σ_u of the operator S_u in the basis $\{|+\rangle, |-\rangle\}$

$$\sigma_u = \sin\theta\cos\varphi\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + \sin\theta\sin\varphi\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} + \cos\theta\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = \begin{pmatrix}\cos\theta & \sin\theta e^{-i\varphi}\\ \sin\theta e^{i\varphi} & \cos\theta\end{pmatrix}$$

2. Calculate the eigenvalues and eigrnvectors of the matrix σ_u

$$\begin{vmatrix}\cos\theta-\lambda & \sin\theta e^{-i\varphi}\\ \sin\theta e^{i\varphi} & -\cos\theta-\lambda\end{vmatrix} = 0 \Rightarrow \lambda^2 = 1$$

For $\lambda = 1$， $\begin{pmatrix}\cos\theta & \sin\theta e^{-i\varphi}\\ \sin\theta e^{i\varphi} & -\cos\theta\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}x\\ y\end{pmatrix}$

$$\begin{cases} x(\cos\theta-1) + y\sin\theta e^{-i\varphi} = 0\\ x\sin\theta e^{i\varphi} - y(1+\cos\theta) = 0\end{cases}$$

$$\frac{x}{y} = \frac{1+\cos\theta}{\sin\theta e^{i\varphi}} = \frac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}e^{i\varphi}} = \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}e^{i\varphi}} \text{ 寫成 } \frac{\cos\frac{1}{2}\theta e^{-i\frac{1}{2}\varphi}}{\sin\frac{1}{2}\theta e^{i\frac{1}{2}\varphi}}$$

$$|+\rangle_u = \cos\frac{1}{2}\theta e^{-i\frac{1}{2}\varphi}|+\rangle + \sin\frac{1}{2}\theta e^{i\frac{1}{2}\varphi}|-\rangle$$

4.4

Prove the formula $(\sigma\bullet A)(\sigma\bullet B) = A\bullet B + i\sigma\bullet(A\times B)$

Where $A = -i\begin{pmatrix}0 & 0\\ I & 0\end{pmatrix}, B = \begin{pmatrix}\sigma & 0\\ 0 & \sigma\end{pmatrix}$

4.5

The two-components Pauli spinors ψ and η satisfy the equations

$\sigma \bullet P\psi + 2im\eta = 0, \sigma \bullet P\eta - iE\psi = 0$

Show that these spinors also satisfy the classcal Schrodinger equation。