Exercises 2.1

Consier a function F(z) the expansion of which in an entire series in z is $F(z)=\sum_{n=0}^{\infty} f_n z^n$ 。

Let A be a operator with which we form the series $F(A)=\sum_{n=0}^{\infty} f_n A^n$ 。

1. Let ψ_a be an eigenfunction of A corresponding to the eigenvalue a， $A\psi_a = a\psi_a$ 。 Prove that ψ_a is also an eigenfunction of F(A) and calculate the corresponding eigenvalue。

$A^n\psi_a = a^n\psi_a$

$$F(A)\psi_a = \sum_{n=0\infty}^{\infty} f_n A^n \psi_a = \sum_{n=0}^{\infty} f_n a^n \psi_a = \psi_a \sum_{n=0}^{\infty} f_n a^n = F(a)\psi_a$$

所以ψ_a是 F(A)的 eigenfunction，其對應的 eigenvalue 為 F(a)。

2. Prove that the matrix of an operator A is diagonal in the orthonormal basis of its eigenfunctions。

設$\{e_1, e_2, ..., e_n\}$ is an orthonormal basis， $Ae_i = \lambda_i e_i$ ， $A=(a_{ij})$

The matrix representation of A in this basis is given by

$a_{ij} = <e_i, Ae_j> = <e_i, \lambda_j e_j> = \lambda_j \delta_{ij}$

3. Consider the following Pauli matrix $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ 。Taking into account the result of Exercise 2，calculate the matrix $\exp(\sigma_z)$ 。

即$A=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ ，calculate exp(A)

$\exp(A)=\sum_{n=0}^{\infty}\frac{A^n}{n!}=\begin{pmatrix} e & 0 \\ 0 & e^{-1} \end{pmatrix}$ ，所以$\exp(\sigma_z)=\begin{pmatrix} e & 0 \\ 0 & e^{-1} \end{pmatrix}$

Exercises 2.2

Let $A(\alpha)$ be an operator which depends on the parameter α 。

By definition the derivative of $A(\alpha)$ with respect to α is the operator given by the limit under the condition that there exists $\frac{dA}{d\alpha} = \lim_{\Delta\alpha \to 0} \frac{A(\alpha+\Delta\alpha)-A(\alpha)}{\Delta\alpha}$

1. Calculate the derivative of the operator $\exp(\alpha A)$ ，where A is an operator which does not depend on α

$e^{\alpha A}=\sum_{n=0}^{\infty}\frac{(\alpha A)^n}{n!}\quad \frac{d(e^{\alpha A})}{d\alpha}=Ae^{\alpha A}$

2. For the moment let $A(\alpha)$ be an operator which depends on α。Calculate the matrix elements of the matrix representing the operator $\frac{dA}{d\alpha}$ as a function of the matrix elements A_{ij} of the matrix M(A)。

Exercises 2.3

Let A be a square matrix of order n and with elements a_{ij} 。The matrix elements of A^k will be denoted $a_{ij}^{(k)}$ 。

1. From the evident majorization $\left|a_{ij}\right|\le M$ ，for any i and j，deduce a majorzation for $a_{ij}^{(2)},a_{ij}^{(3)}$ ，and then for $a_{ij}^{(k)}$

The elements of A^2 are given by $a_{ij}^2=\sum_{k=1}^{n}a_{ik}a_{kj}$ 。The majorziation $\left|a_{ij}\right|\le M$ implies

$\left|a_{ij}^{(2)}\right|\le\sum_{k=1}^{n}\left|a_{ik}\right|\left|a_{kj}\right|\le nM^2$

Similarly，$\left|a_{ij}^{(3)}\right|\le n^2M^3$ ，by induction $\left|a_{ij}^{(k)}\right|\le n^{k-1}M^k$

2. If the n^2 numerical series $S_{ij}=\delta_{ij}+\frac{a_{ij}}{1!}+\frac{a_{ij}^{(2)}}{2!}+...+\frac{a_{ij}^{(k)}}{k!}+...$ are convergent，it will be said the matrix series $\exp(A)=I+\frac{A}{1!}+\frac{A^2}{2!}+...+\frac{A^k}{k!}+...$converges，and the matrix formed by the sum of this series will be written exp(A)。Using the majorization of the $a_{ij}^{(k)}$ ，show that the series denoted exp(A) converges for every matrix A。

3. Show that a sufficient condition for exp(A)exp(B)=exp(A+B) to hold is that the matrices A and B commute。

Exercises 2.4

A two-dimensional representation of the group of plane rotation SO(2) is given by the matrices $M(R_\alpha)=\begin{pmatrix}\cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha\end{pmatrix}$

1. Calculate the infinitesimal matrix $X^{(\alpha)}$ of the given representation

Differentiation of the matrix elements of $M(R_\alpha)$ at $\alpha=0$ gives the infinitesmal matrix

$X^{(\alpha)}=\begin{pmatrix}0 & -1 \\ 1 & 0\end{pmatrix}$

2. By a direct calculation verify the following expression for the matrices $M(R_\alpha)$ of this representation $M(R_\alpha)=\exp(\alpha X^{(\alpha)})$

$X^{(\alpha)}=\begin{pmatrix}0 & -1 \\ 1 & 0\end{pmatrix}$，計算 $(X^{(\alpha)})^{2n},(X^{(\alpha)})^{2n+1}$

The matrix $\exp(\alpha X^{(\alpha)}):=I+\dfrac{\alpha X^{(\alpha)}}{1!}+\dfrac{\alpha^2(X^{(\alpha)})^2}{2!}+\ldots+\dfrac{\alpha^k(X^{(\alpha)})^k}{k!}+\ldots$ 拆開成兩部分

$=I\cos\alpha+X^{(\alpha)}\sin\alpha=\begin{pmatrix}\cos\alpha & 0 \\ 0 & \cos\alpha\end{pmatrix}+\begin{pmatrix}0 & -\sin\alpha \\ \sin\alpha & 0\end{pmatrix}=\begin{pmatrix}\cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha\end{pmatrix}$

Exercises 2.5

The matrices $M(R_\alpha)$ of the representation of the group SO(2) are related by the relation

$M(R_\alpha)X^{(\alpha)}=\dfrac{dM(R_\alpha)}{d\alpha}$ with $X^{(\alpha)}=\begin{pmatrix}0 & -1 \\ 1 & 0\end{pmatrix}$

Calculate the matrices $M(R_\alpha)$ by solving the differential system which the elements M_{ij} of these matrices satisfy

$$\begin{pmatrix}M_{11} & M_{12} \\ M_{21} & M_{22}\end{pmatrix}\begin{pmatrix}0 & -1 \\ 1 & 0\end{pmatrix}=\begin{pmatrix}M_{12} & -M_{11} \\ M_{22} & -M_{21}\end{pmatrix}=\begin{pmatrix}\dfrac{dM_{11}}{d\alpha} & \dfrac{dM_{12}}{d\alpha} \\ \dfrac{dM_{21}}{d\alpha} & \dfrac{dM_{22}}{d\alpha}\end{pmatrix} \text{ with } M(R(0))=I$$

The initial condition $M_{11}(0)=M_{22}(0)=1, M_{12}(0)=M_{21}(0)=0$

Then we obtain $M_{11}=M_{22}=\cos\alpha, M_{12}=-\sin\alpha, M_{21}=\sin\alpha$

Exercises 2.6

1. By expanding the expression $\exp(-i\frac{1}{2}\alpha\sigma_z)$，where $\sigma_z=\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}$，prove that a second-order matrix representation an element of the group SU(2) is obtained。

The matrix $\exp(-i\frac{1}{2}\alpha\sigma_z)$ has as its expansion in a series

$$\exp(-i\frac{1}{2}\alpha\sigma_z) = I + \sum_{n=1}^{\infty}(-i\frac{1}{2}\alpha)^n \frac{\sigma_z^n}{n!}$$ ，where $(\sigma_z)^{2n} = I, (\sigma_z)^{2n+1} = \sigma_z$

consequently， $\exp(-i\frac{1}{2}\alpha\sigma_z) = \cos\frac{\alpha}{2} I - i\sin\frac{\alpha}{2}\sigma_z = \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{pmatrix}$

2. Calculate the expression for the spinor $\eta = (\psi_1, \psi_2)$ transformed by the matrix $\exp(-i\frac{1}{2}\alpha\sigma_z)$ 。

   What is the expression for the transgorm for $\alpha = 2\pi$? For which angle of rotation do we obtain an identity spinor ?

A spinor transformed by this matrix becomes $\eta' = \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} e^{-i\alpha/2}\psi_1 \\ e^{i\alpha/2}\psi_2 \end{pmatrix}$

A rotation through an angle of 2π about the axis Oz transforms a spinor into its opposite。It is a rotation through an angle of 4π which once again gives a spinor identical to itself。