§ Exercises

這些是有解答的習作，在 p.29~p.33 [Spinors in Physics]

1.1 Consider a unitary spinor (ψ,ϕ) defined by $\begin{cases} x=\psi\phi^*+\psi^*\phi \\ y=i(\psi\phi^*-\psi^*\phi) \\ z=\psi\psi^*-\phi\phi^* \end{cases}$ ，where (x,y,z) is a point M of three-dimsional space。

(1) Show that the point M(x, y, z) lies on a sphere of unit radius。

As the spinor is unitary we have $\psi\psi^*+\phi\phi^*=1$

$$x^2+y^2+z^2=(\psi\phi^*+\psi^*\phi)^2-(\psi\phi^*-\psi^*\phi)^2+(\psi\psi^*-\phi\phi^*)^2=(\psi\psi^*+\phi\phi^*)^2=1$$

(2) Consider the following transformation U

$\psi'=a\psi+b\phi,\phi'=-b^*\psi+a^*\phi$

Where a and b are complex parameters satisfying $aa^*+bb^*=1$

Show that the matrix M(U) of this transformation is unitary。

A matrix M(U) is said to be unitary if $M(U)M(U)^\dagger=M(U)^\dagger M(U)=I$

$M(U)=\begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$ As the adjoint matrix is the transpose of the congugate(共軛後轉置)，we have $M(U)^\dagger=\begin{pmatrix} a^* & -b \\ b^* & a \end{pmatrix}$，

it is easy to verify that $M(U)M(U)^\dagger=M(U)^\dagger M(U)=I$

(3) Show that the transformation U takes a point M(x.y.z) of the unit sphere into another point $M'(x',y',z')$ of the same sphere

$\psi\psi^*+\phi\phi^*=1$ and $aa^*+bb^*=1$，then

$$\psi'\psi'^*+\phi'\phi'^*=(a\psi+b\phi)(a\psi+b\phi)^*+(-b^*\psi+a^*\phi)(-b^*\psi+a^*\phi)^*=...=1$$

$$x'^2+y'^2+z'^2=(\psi'\psi'^*+\phi'\phi'^*)^2=1$$

(4) Show that the transformation which takes the point M(x,y,z) into the point $M'(x',y',z')$ on the unit sphere is a rotation R in three-dimensional place。

$\begin{cases} x'=\psi'\phi'^*+\psi'^*\phi' \\ y'=i(\psi'\phi'^*-\psi'^*\phi') \\ z'=\psi'\psi'^*-\phi'\phi'^* \end{cases}$， $x'^2+y'^2+z'^2=(\psi'\psi'^*+\phi'\phi'^*)^2=1$， $\psi\psi^*+\phi\phi^*=1$

The transformation on the coordinates is linear and norm-preserving，which corresponds to a rotation in three-dimensional space。

This is further supported by the fact that the spinor transformation is an element of SU(2)，and there exists a homomorphism from SU(2) to SO(3)，the group of rotations in three dimensions。

Thus，the transformation that takes pointM(x,y,z) to $M'(x', y', z')$ is a rotation.

(5) Show that two transformations U ans －U correspond to every rotation R in three-dimensional space。

The point $M = (x, y, z)$ is defined in terms of the spinor (ψ, ϕ) as:

$$x = \psi\phi^* + \psi^*\phi, \quad y = i(\psi\phi^* - \psi^*\phi), \quad z = \psi\psi^* - \phi\phi^*.$$

This can be expressed using the Pauli matrices $\sigma_x, \sigma_y, \sigma_z$ as:

$$M = S^\dagger \sigma S,$$

where $S = \begin{pmatrix} \psi \\ \phi \end{pmatrix}$ and $\sigma = (\sigma_x, \sigma_y, \sigma_z)$.

The transformation U is given by:

$$\psi' = a\psi + b\phi, \quad \phi' = -b^*\psi + a^*\phi,$$

with a and b complex parameters satisfying $aa^* + bb^* = 1$. This defines a unitary matrix $U = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$ in $\mathrm{SU}(2)$, since $\det U = 1$ and $UU^\dagger = I$.

After applying U, the new spinor is $S' = US$, and the new point $M' = (x', y', z')$ is:

$$M' = S'^\dagger \sigma S' = (US)^\dagger \sigma (US) = S^\dagger U^\dagger \sigma U S.$$

For $U \in \mathrm{SU}(2)$, the conjugation $U^\dagger \sigma U$ corresponds to a rotation R in $\mathrm{SO}(3)$ such that:

$$U^\dagger \sigma U = R\sigma,$$

where R is a 3×3 rotation matrix. Thus:

$$M' = S^\dagger R\sigma S = R(S^\dagger \sigma S) = RM,$$

so M' is the rotated vector.

If U is replaced with $-U$, then:

$$(-U)^\dagger \sigma (-U) = U^\dagger \sigma U,$$

since the minus signs cancel. Therefore, both U and $-U$ yield the same rotation R.

Thus, for every rotation R in three-dimensional space, there are two spinor transformations U and $-U$ that correspond to it.

1.2 The lines with coefficient equal to $\pm i$ of a plane referred to two rectangular axes are called isotropic lines

在狹義相對論中，在時空度量 $ds^2 = c^2t^2 - x^2 - y^2 - z^2$ 下，若一條世界線滿足 $ds^2 = 0$ 那就是 isotropic line。(又稱 null line，lightlike line。) 表示「光子」的運動路徑，其自身的時空間隔為零。

(1) Write down the equations of the isotropic lines in a plane xOy。

空間	二次形式 $Q(x,y)$	isotropic line 方程式
一般情形	$ax^2 + 2bxy + cy^2$	$a + 2bk + ck^2 = 0$
歐氏平面	$x^2 + y^2 = 0$	$y = \pm ix$
洛侖茲平面	$x^2 - y^2 = 0$	$y = \pm x$

(2) Show that the angular coefficient of an isotropic line is invariant under every change of axes of rectangular coordinates。

(3) Let $M_1 = (x_1, y_1), M_2(x_2, y_2)$ be two points of the same isotropic line。Show that the distance between these points is zero。

(4) Let a vector X=(x,y) lie along an isotropic line。Determine the length of X。

1.3 Acoording to relations $\begin{cases} x = \psi\phi^* + \psi^*\phi \\ y = i(\psi\phi^* - \psi^*\phi) \\ z = \psi\psi^* - \phi\phi^* \end{cases}$，the vector $\overrightarrow{OP} = (x, y, z)$

Using the components of the spinors (ψ, ϕ) and (ψ^*, ϕ^*) put into matrix form, write down the components of the vector $\boldsymbol{OP}$ in the form of products of matrices using the Pauli matrices.