Exercise 6.1
To the four-vector $\boldsymbol{X} = (x^0, x^1, x^2, x^3)$ of Minkowski space we make correspond the matrix $\chi = x^i \sigma_i$, where σ_i denotes the Pauli matrices.

1. Write the matrix χ explicitly and calculate the inverse correspondence.
2. Calculate the scalar product $(\boldsymbol{X}, \boldsymbol{X})$ as a function of χ.
3. Let A be a matrix belonging to the group $\boldsymbol{SL}(2, \mathbb{C})$. Determine a transformation which preserves the scalar product of four-vectors.
4. From this deduce that there exists a Lorentz transformation Λ_A corresponding to the matrix A.
5. Show that there exists a homomorphism between the groups $\boldsymbol{SL}(2, \mathbb{C})$ and $\boldsymbol{SO}(3, 1)^{\dagger}$.

Exercise 6.2
The infinitesimal matrices Y_{ij} of the group $\boldsymbol{SO}(3, 1)^{\dagger}$ are given by (6.1.9).

1. Calculate the commutators $[Y_{02}, Y_{01}]$, $[Y_{12}, Y_{01}]$, $[Y_{02}, Y_{12}]$.
2. Compare them with the commutation relations of $\boldsymbol{SO}(3)$.

Exercise 6.3
Consider two spinors (ψ^1, ψ^2) and (ϕ^1, ϕ^2) of four-dimensional space.

1. Show that the product $(\psi^1 \phi^2 - \psi^2 \phi^1)$ is invariant under all transformations of the group $\boldsymbol{SL}(2, \mathbb{C})$.
2. In order that the product $(\psi^1 \phi^2 - \psi^2 \phi^1)$ may be written in the classical form $\psi^k \phi_k$ of a scalar product, determine the expression for the 'covariant' components ϕ_k as well as the components of the fundamental tensor.

Exercise 6.4
Consider two Cartesian frames of reference $Oxyz(t)$ and $O'x'y'z'(t')$ which are moving with respect to each other at a speed v along the axis Ox parallel to $O'x'$.

1. Recall the formulas for the Lorentz transformation between two reference frames. Set

$$\beta = \frac{v}{c}.$$

2. For the moment use the following notations

$$\begin{aligned} ct &= x^0, & x &= x^1, & y &= x^2, & z &= x^3, \\ ct' &= x^{0\prime} & x' &= x^{1\prime}, & y' &= x^{2\prime}, & z' &= x^{3\prime}. \end{aligned} \tag{6.4.26}$$

Set

$$\gamma = (1 - \beta^2)^{-1/2}.$$

Calculate the matrix of the Lorentz transformation between the two reference frames.

3. Show that we can, indeed, set

$$\gamma = \cosh\varphi, \qquad \gamma\beta = \sinh\varphi.$$