Eigenvalues for a Schrödinger operator on a closed Riemannian manifold with holes

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Abstract

In this article we consider a closed Riemannian manifold (M, g) and A a subset of M. The purpose of this article is the comparison between the eigenvalues $(\lambda_k(M))_{k\geq 1}$ of a Schrödinger operator $P := -\Delta_g + V$ on the manifold (M, g) and the eigenvalues $(\lambda_k(M - A))_{k\geq 1}$ of P on the manifold $(M - A, g)$ with Dirichlet boundary conditions.

1 Introduction

The behaviour of the spectrum of a Riemannian manifold (M, g) under topological perturbation has been the subject of many research. The most famous exemple is the crushed ice problem **[Kac]**, see also **[Ann]**. This problem consists to understand the behaviour of Laplacian eigenvalues with Dirichlet boundary on a domain with small holes. This subject was first studied by M. Kac **[Kac]** in 1974. Then, J. Rauch and M. Taylor **[Ra-Ta]** studied the case of Euclidian Laplacian in a compact set M of $\mathbb{R}^n$: they showed that the spectrum of $\Delta_{\mathbb{R}^n}$ is invariant by a topological excision of a M by a compact subset A with a Newtonian capacity zero. Later, S. Osawa, I. Chavel and E. Feldman **[Ca-Fe1]**, **[Ca-Fe2]** treated the Riemmannian manifold case. They used complex probalistic techniques based on Brownian motion. In **[Ge-Zh]**, F. Gesztesy and Z. Zhao investigate the study the case of a Schrödinger operator with Dirichlet boundary conditions $\mathbb{R}^n$, they use probabilistic tools. In 1995, in a nice article **[Cou]** G . Courtois studied the case of Laplace Beltrami operator on closed Riemannian manifold. He used very simple techniques of analysis. In **[Be-Co]** J. Bertrand and B. Colbois explained also the case of Laplace Beltrami operator on compact Riemannian manifold. In this article we focus on the the Schrödinger operator $-\Delta_g + V$ case on a closed Riemannian manifold.

Assumption. *The manifold is closed (i.e. compact without boundary); the function V is bounded on the manifold M and* $\min_M V > 0$.

In this work we show that under "little" topological excision of a part A from the manifold, the spectrum of $-\Delta_g + V$ on $M - A$ is close of the spectrum on M. More precisely, the "good" parameter for measuring the littleness of A is a type of electrostatic capacity defined by :

$$\operatorname{cap}(A) := \inf \left\{ Q(u),\, u \in H^1(M),\, \int_M u \, d\mathcal{V}_g = 0,\, u - e_1 \in H_0^1(M - A) \right\}$$

where e_1 denotes the first eigenfunction of the operator $-\Delta_g + V$ on the manifold M, and Q is the following quadratic form :

$$Q(\varphi) := \int_M |d\varphi|^2 \, d\mathcal{V}_g + \int_M V \, |\varphi|^2 \, d\mathcal{V}_g$$

and $H_0^1(M-A)$ is the *Sobolev space* defined by :

$$H_0^1(M-A) := \overline{\{g \in H^1(M),\, g = 0 \text{ on a open neighborhood of } A\}}$$

the closure is for the norm $\|.\|_{H^1(M)}$, $H^1(M)$ is the usual Sobolev space on M. Indeed, more cap(A) is small, more the spectrum $-\Delta_g + V$ on $M - A$ is close of the spectrum on M in the following sense :

Theorem. *Let (M, g) a closed Riemannian manifold. For all integer $k \geq 1$, there exists a constant C_k depending on the manifold (M, g) and on the potential V such that for all subset A of M we have :*

$$0 \leq \lambda_k(M-A) - \lambda_k(M) \leq C_k \sqrt{cap(A)}.$$

The organization of this paper is the following : in the part 2 we start by recall some classicals results in spectral theory and about usual Sobolev spaces, next we define our specific Sobolev space $H_0^1(M-A)$ and the notion of Schrödinger capacity. In particular, we explain the link between the functionnal Hilbert space $H_0^1(M-A)$ and Schrödinger capacity cap(A). The last part of this paper is a detailed proof of the main theorem.

2 Spectral problem background

2.1 Schrödinger operator on a Riemannian manifold

We recall here some generality on spectral geometry. In Riemannian geometry, the *Laplace Beltrami operator* is the generalisation of Laplacian $\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$ on $\mathbb{R}^n$. For a $\mathcal{C}^2$ real valued function f on a Riemannian manifold and for a local chart $\phi : U \subset M \to \mathbb{R}$ of the manifold M, the Laplace Beltrami operator is given by the local expression :

$$\Delta_g f = \frac{1}{\sqrt{g}} \sum_{j,k=1}^{n} \frac{\partial}{\partial x_j} \left(\sqrt{g} g^{jk} \frac{\partial (f \circ \phi^{-1})}{\partial x_k} \right) \tag{2.1}$$

where $g = \det(g_{ij})$ and $g^{jk} = (g_{jk})^{-1}$.
The spectrum of this operator is a nice geometric invariant, see Berger, Gauduchon and Mazet **[BGM]** and **[Bé-Be]**. The spectrum of Laplace Beltrami operator has many applications in geometry topology, physics ,etc ...
For every Riemannian manifold (M, g) with dimension $n \geq 1$ we have the "natural" Hilbert space $L^2(M) = L^2(M, d\mathcal{V}_g)$, $\mathcal{V}_g$ is the Riemannian volume form associated to the metric g. For V a function from M to $\mathbb{R}$, we define the Schrödinger operator on the manifold (M, g) by the linear unbounded operator on the set of smooth compact supports real valued functions $\mathcal{C}_c^\infty(M) \subset L^2(M)$ by : $-\Delta_g + V$.

2.2 Sobolev spaces

Let us denotes by $\mathcal{C}_c^\infty(M)$ the set of smooth functions with compact support in M. The set $\mathcal{C}_c^\infty(M)$ is also called the set of *test functions* in the language of distributions. Recall first that the *Lebesgue space* $L^2(M)$ on the manifold (M, g) is defined by :

$$L^2(M) := \left\{ f \,:\, M \to \mathbb{R} \text{ measurable such that } \int_M |f|^2 \, d\mathcal{V}_g < +\infty \right\}.$$

This space is a Hilbert space for the scalar product :

$$\langle u, v \rangle_{L^2} := \int_M uv \, d\mathcal{V}_g.$$

Next the *Sobolev space* $H^1(M)$ is defined by :

$$H^1(M) := \overline{\mathcal{C}^\infty(M)}$$

where the closure is for the norm $\|.\|_{H^1}$: $\|u\|_{H^1} := \sqrt{\|u\|_{L^2}^2 + \|du\|_{L^2}^2}$.
An other point of view to define the space $H^1(M)$ is the following :

$$H^1(M) = \left\{ u \in L^2(M);\, du \in L^2(M) \right\}$$

where the derivation is the sense of distribution.
The space $H^1(M)$ is a Hilbert space for the scalar product :

$$\langle u, v \rangle_{H^1} := \langle u, v \rangle_{L^2} + \langle du, dv \rangle_{L^2}.$$

For finish, the Sobolev space $H_0^1(M, g)$ is defined by :

$$H_0^1(M) := \overline{\mathcal{C}_c^\infty(M)}$$

the closure is for the norm $\|.\|_{H^1(M)}$.
So we have :

$$\mathcal{C}_c^\infty(M) \subset H_0^1(M) \subset H^1(M) \subset L^2(M).$$

Recall that, for the norm $\|.\|_{L^2(M)}$ we have :

$$\overline{\mathcal{C}_c^\infty(M)} = L^2(M).$$

2.3 Spectral problem

The spectral problem is the following : find all pairs (λ, u) with $\lambda \in \mathbb{R}$ and $u \in L^2(M)$ such that :

$$-\Delta_g u + Vu = \lambda u \tag{2.2}$$

(with $u \in L^2(M)$ in the non-compact case).

In the case of manifold with boundary, we need boundary conditions on the functions u, for example the Dirichlet conditions : $u = 0$ on the boundary of M, or Neumann conditions : $\frac{\partial u}{\partial n} = 0$ on the boundary of M. In the case of closed manifolds (compact without boundary) we don't have conditions.
For our context (the closed case) the natural space to look here is the Sobolev space $H^1(M)$.
Recall here a classical theorem of spectral theory (see for example **[Re-Si]**) :

Theorem. *For the above problems, the operator $-\Delta_g + V$ is self-adjoint, the spectrum of the operator $-\Delta_g + V$ consists of a sequence of infinite increasing eigenvalues with finite multiplicity :*

$$\lambda_1(M) \leq \lambda_2(M) \leq \cdots \leq \lambda_k(M) \leq \cdots \to +\infty.$$

Moreover, the associate eigenfunctions $(e_k)_{k\geq 0}$ is a Hilbert basis of the space $L^2(M)$.

Definition. We define the quadradic form Q with domain $D(Q) := H^1(M)$ by :

$$Q(\varphi) := \int_M |d\varphi|^2 \, d\mathcal{V}_g + \int_M V \, |\varphi|^2 \, d\mathcal{V}_g.$$

Recall also (see for example **[Co-Hi]**) the minimax variational characterization for eigenvalues : for all $k \geq 1$

$$\lambda_k(M) = \min_{\substack{E\subset H^1(M)\\ \dim(E)=k}} \max_{\substack{\varphi\in E\\ \varphi\neq 0}} R(\varphi) \tag{2.3}$$

where $R(\varphi)$ is the *Rayleigh quotient* of the function φ :

$$R(\varphi) := \frac{Q(\varphi)}{\int_M \varphi^2 \, d\mathcal{V}_g}. \tag{2.4}$$

In our context, a consequence of the minimax principle is :

Proposition. *The first eigenvalue $\lambda_1(M)$ and e_1 the first eigenfunction of the operator $-\Delta_g + V$ on the manifold (M, g) satisfy $\lambda_1(M) \geq \min_M V > 0$ and $e_1 > 0$ or $e_1 < 0$ in M.*

Proof. It's clear that

$$\int_M |de_1|^2 \, d\mathcal{V}_g + \int_M V \, |e_1|^2 \, d\mathcal{V}_g \geq \min_M V \, \|e_1\|^2_{L^2(M)}$$

and on the other hand

$$\int_M |de_1|^2 \, d\mathcal{V}_g + \int_M V \, |e_1|^2 \, d\mathcal{V}_g = -\int_M \Delta_g e_1 e_1 \, d\mathcal{V}_g + \int_M V \, |e_1|^2 \, d\mathcal{V}_g$$

$$= \int_M \left(-\Delta_g + V\right) e_1 e_1 \, d\mathcal{V}_g = \lambda_1(M) \, \|e_1\|^2_{L^2(M)}$$

so $\lambda_1(M) \geq \min_M V$. Next, suppose the function e_1 changes sign into M, since $e_1 \in H^1(M)$, the function $f := |e_1|$ belongs to $H^1(M)$ and $|df| = |de_1|$ (see for example **[Gi-Tr]**), hence $R(f) = R(e_1)$. So, the function f is a first eigenfunction of $-\Delta_g + V$ on the manifold M which satisfies $f \geq 0$ on M, f vanish into M and $\left(-\Delta_g + V\right) f = \lambda_1(M) f \geq 0$ on M. Using the maximum principle **[Pr-We]**, the function f can not achieved it minimum in an interior point of the manifold M, hence f does not vanish on M, so we obtain a contradiction. □

3 Proof of the main theorem

3.1 Somes other usefull spaces

We define on the space $H^1(M)$ the $\star$-norm by :

$$\|u\|_\star^2 := \int_M |du|^2 \, d\mathcal{V}_g + \int_M V\,|u|^2 \, d\mathcal{V}_g$$

so, without difficulty we have :

Proposition. *The application $\|.\|_\star$ is a norm on the space $H^1(M)$; moreover this norm is equivalent to the Sobolev norm $\|.\|_{H^1(M)}$. In particular $H^1(M), \|.\|_\star$ is a Banach space.*

Let us denotes by $\mathcal{C}_c^\infty(M-A)$ the set of smooth functions with compact support on $M-A$. For a compact subset A of the manifold M the usual Sobolev space $H_0^1(M-A)$ is defined by the closure of $\mathcal{C}_c^\infty(M-A)$ for the norm $\|.\|_{H^1(M)}$:

$$H_0^1(M-A) := \overline{\mathcal{C}_c^\infty(M-A)}.$$

What happens when the set A is not compact ? For example if A is a dense and countable subset of points of the manifold M, the space of test functions $\mathcal{C}_c^\infty(M-A)$ is reduced to $\{0\}$. Therefore we cannot define the space $H_0^1(M-A)$. In this case, we propose a definition of $H_0^1(M-A)$ for any subset A of M.

Definition. We define the Sobolev spaces $\mathcal{H}_0^1(M-A)$ and $H_0^1(M-A)$ by :

$$\mathcal{H}_0^1(M-A) := \Big\{ g \in H^1(M),\ g = 0 \text{ on a open neighborhood of } A \Big\}\,;$$

$$H_0^1(M-A) := \overline{\mathcal{H}_0^1(M-A)}$$

where the closure is for the norm $\|.\|_{H^1(M)}$.

We have the :

Proposition. *If the set A is compact, the previous definition of the space $H_0^1(M-A)$ coincides with the usal ones.*

Proof. Let $f \in H_0^1(M-A) := \overline{\mathcal{H}_0^1(M-A)}$, then by definition : for all $\varepsilon \geq 0$ there exists $g \in \mathcal{H}_0^1(M-A)$ such that $\|f-g\|_{H^1(M)} \leq \varepsilon$. So, we will show that we can write g as a limit of sequence from the space $\mathcal{C}_c^\infty(M-A)$ and conclude. Since $g \in \mathcal{H}_0^1(M-A)$ there exists an open set $U \supset A$ such that $g_{|U} = 0$. Consider two open sets U_1 and U_2 of the manifold M such that :

$$A \subset U_1,\ M-U \subset U_2,\ U_1 \cap U_2 = \emptyset;$$

and consider also a function $\varphi \in \mathcal{D}(M)$ such that :

$$\varphi_{|U_1} = 0,\ \varphi_{|U_2} = 1.$$

Of course, the function φ belongs to the space $\mathcal{C}_c^\infty(M-A)$. Next, since $g \in \mathcal{H}_0^1(M-A) \subset H^1(M)$ and as the set of smooth functions $\mathcal{C}^\infty(M)$ is dense in

$H^1(M)$: there exists a sequence $(g_n)_n$ in $\mathcal{C}^\infty(M)$ such that $\lim_{n\to+\infty} g_n = g$ for the norm $\|.\|_{H^1(M)}$. Therefore we claim that : $\lim_{n\to+\infty} \varphi g_n = g$ for the norm $\|.\|_{H^1(M)}$. Indeed, start by, for all integer n :

$$\begin{aligned} \|\varphi g_n - g\|^2_{H^1(M)} &\leq \|g_n - g\|^2_{H^1(M-U)} + \|\varphi g_n - g\|^2_{H^1(U)} \\ &\leq \|g_n - g\|^2_{H^1(M)} + \|\varphi g_n - g\|^2_{H^1(U)} . \end{aligned}$$

Next, we observe that, for all integer n :

$$\begin{aligned} \|\varphi g_n - g\|^2_{H^1(U)} &= \|\varphi g_n\|^2_{H^1(U)} \\ &= \int_U |\varphi g_n|^2 \, d\mathcal{V}_g + \int_U |d\varphi g_n + \varphi d g_n|^2 \, d\mathcal{V}_g \\ &\leq \int_U |\varphi g_n|^2 \, d\mathcal{V}_g + \int_U |d\varphi g_n|^2 \, d\mathcal{V}_g + \int_U |\varphi d g_n|^2 \, d\mathcal{V}_g + 2\int_U |d\varphi g_n \varphi d g_n| \, d\mathcal{V}_g \\ &\leq \|\varphi\|^2_\infty \|g_n\|^2_{L^2(U)} + \|d\varphi\|^2_{L^\infty(M)} \|g_n\|^2_{L^2(U)} \\ &\quad + \|\varphi\|^2_\infty \|dg_n\|^2_{L^2(U)} + 2\|d\varphi\|_\infty \|\varphi\|_\infty \int_U |g_n d g_n| \, d\mathcal{V}_g \\ &\leq \|\varphi\|^2_\infty \|g_n\|^2_{L^2(U)} + \|d\varphi\|^2_\infty \|g_n\|^2_{L^2(U)} \\ &\quad + \|\varphi\|^2_\infty \|dg_n\|^2_{L^2(U)} + 2\|d\varphi\|_\infty \|\varphi\|_{L^\infty(M)} \|g_n\|_{L^2(U)} \|dg_n\|_{L^2(U)} , \end{aligned}$$

by Cauchy-Schwarz inequality.
Finally we get for all integer n :

$$\|\varphi g_n - g\|^2_{H^1(U)} \leq \|g_n\|^2_{H^1(U)} \left(2\|\varphi\|^2_\infty + \|d\varphi\|^2_\infty + 2\|d\varphi\|_\infty \|\varphi\|_\infty\right) .$$

As a consequence, we have for all integer n :

$$\begin{aligned} \|\varphi g_n - g\|^2_{H^1(M)} &\leq \|g_n - g\|^2_{H^1(M-U)} \\ &\quad + \|g_n\|^2_{H^1(U)} \left(2\|\varphi\|^2_\infty + \|d\varphi\|^2_\infty + 2\|d\varphi\|_\infty \|\varphi\|_\infty\right) . \end{aligned}$$

Now, it suffices to note that $\|g_n\|^2_{H^1(U)} = \|g_n - g\|^2_{H^1(U)} \leq \|g_n - g\|^2_{H^1(M)}$ (since $g = 0$ on the open set U) and we have finally :

$$\|\varphi g_n - g\|^2_{H^1(M)} \leq$$
$$\|g_n - g\|^2_{H^1(M)} \left(1 + 2\|\varphi\|^2_\infty + \|d\varphi\|^2_\infty + 2\|d\varphi\|_\infty \|\varphi\|_\infty\right) .$$

The sequence $(\varphi g_n)_n$ belong to $\mathcal{C}^\infty_c(M-A)^{\mathbb{N}}$,and since $\lim_{n\to+\infty} g_n = g$ for the norm $\|.\|_{H^1(M)}$ the previous inequality implies $\lim_{n\to+\infty} \varphi g_n = g$ for the norm $\|.\|_{H^1(M)}$.
So we have shown that every function $f \in H^1_0(M-A) := \overline{\mathcal{H}^1_0(M-A)}$ is a limit (for the norm $\|.\|_{H^1(M)}$) of a sequence of $\mathcal{C}^\infty_c(M-A)$.
Conversely, since $\mathcal{C}^\infty_c(M-A) \subset \mathcal{H}^1_0(M-A)$ we get :

$$H^1_0(M-A) := \overline{\mathcal{C}^\infty_c(M-A)} \subset H^1_0(M-A) := \overline{\mathcal{H}^1_0(M-A)}.$$

□

Let us also denote the spaces $H^1_\star(M)$ and $S_A(M)$ by :

$$H^1_\star(M) := \left\{ f \in H^1(M),\ \int_M f\, d\mathcal{V}_g = 0 \right\};$$

and

$$S_A(M) := \left\{ u \in H^1_\star(M),\ u - e_1 \in H^1_0(M - A) \right\}.$$

In the definition of the space $H^1_\star(M)$ the condition $\int_M f\, d\mathcal{V}_g = 0$ is analog to a boundary condition. We observe that the space $H^1_\star(M)$ is a Hilbert space for the norm :

$$\|u\|_\star := \int_M |du|^2\, d\mathcal{V}_g + \int_M V\, |u|^2\, d\mathcal{V}_g;$$

and $S_A(M)$ is just an affine closed subset of $H^1(M)$.

3.2 Schrödinger capacity

Next, we introduce the Schrödinger capacity of the set A ;

Definition. Let us consider the *Schrödinger capacity* $\mathrm{cap}(A)$ of the set A defined by

$$cap(A) := inf \left\{ \int_M |du|^2\, d\mathcal{V}_g + \int_M V\, |u|^2\, d\mathcal{V}_g,\ u \in S_A(M) \right\}. \tag{3.1}$$

Let us remark that : there exists an unique function $u_A \in S_A(M)$ such that

$$\mathrm{cap}(A) = \int_M |du_A|^2\, d\mathcal{V}_g + \int_M V\, |u_A|^2\, d\mathcal{V}_g.$$

Indeed : here the capacity $\mathrm{cap}(A)$ is just the distance between the function 0 and the closed space $S_A(M)$. This distance is equal to $\|u_A\|_\star$ where u_A is the orthogonal projection of 0 on $S_A(M)$:

$$cap(A) = d_\star\,(0, S_A(M)) := \inf\left\{ \|u\|_\star,\ u \in S_A(M) \right\} = \|u_A\|_\star.$$

In the following lemma we give the relationships between the capacity $\mathrm{cap}(A)$, the functions u_A, e_1 and the Sobolev spaces $H^1_0(M - A)$, $H^1(M)$.

Lemma. *For all subset A of the manifold M, the following properties are equivalent :*
(i) $cap(A) = 0$;
(ii) $u_A = 0$;
(iii) $e_1 \in H^1_0(M - A)$;
(iv) $H^1_0(M - A) = H^1(M)$.

Proof. It is clear from the formula (3.1) that $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$. Next, suppose the property (iii) holds : so there exists a sequence $(v_n)_n \in \mathcal{H}^1_0(M - A)^{\mathbb{N}}$ such that $\lim_{n\to+\infty} v_n = e_1$ for the norm $\|.\|_{H^1(M)}$. So, for all smooth function $\varphi \in \mathcal{C}^\infty(M)$ we have $\lim_{n\to+\infty} (\varphi v_n)/e_1 = \varphi$ for the norm $\|.\|_{H^1(M)}$, indeed for all integer n :

$$\left\| \frac{\varphi v_n}{e_1} - \varphi \right\|^2_{H^1(M)} = \int_M \left| \frac{\varphi v_n}{e_1} - \varphi \right|^2 d\mathcal{V}_g + \int_M \left| d\left(\frac{\varphi v_n}{e_1} \right) - d\varphi \right|^2 d\mathcal{V}_g.$$

First, we have for all integer n :

$$\int_M \left| \frac{\varphi v_n}{e_1} - \varphi \right|^2 d\mathcal{V}_g = \int_M \frac{1}{|e_1|^2} |\varphi (v_n - e_1)|^2 d\mathcal{V}_g$$

$$\leq \left\| \frac{1}{e_1} \right\|_\infty^2 \|\varphi\|_\infty^2 \|v_n - e_1\|_{L^2(M)}^2$$

so, since $\lim_{n\to+\infty} v_n = e_1$ for the norm $\|.\|_{H^1(M)}$ we have

$$\lim_{n\to+\infty} \int_M \left| \frac{\varphi v_n}{e_1} - \varphi \right|^2 d\mathcal{V}_g = 0.$$

On the other hand, for all integer n :

$$\int_M \left| d\left(\frac{\varphi v_n}{e_1}\right) - d\varphi \right|^2 d\mathcal{V}_g = \int_M \left| \frac{d(\varphi v_n) e_1 - \varphi v_n d e_1}{e_1^2} - d\varphi \right|^2 d\mathcal{V}_g$$

$$= \int_M \left(\frac{1}{e_1^2}\right) \left| d(\varphi) v_n e_1 + \varphi d(v_n) e_1 - \varphi v_n d(e_1) - d(\varphi) e_1^2 \right|^2 d\mathcal{V}_g$$

$$\leq \left\| \frac{1}{e_1} \right\|_\infty^2 \left\| d\varphi v_n e_1 - d\varphi e_1^2 + \varphi d v_n e_1 - \varphi v_n d e_1 \right\|_{L^2(M)}^2$$

$$\leq \left\| \frac{1}{e_1} \right\|_\infty^2 \left(\left\| d\varphi v_n e_1 - d\varphi e_1^2 \right\|_{L^2(M)} + \|\varphi d v_n e_1 - \varphi v_n d e_1\|_{L^2(M)} \right)^2$$

$$\leq \left\| \frac{1}{e_1} \right\|_\infty^2 \Big[\|d\varphi\|_\infty \|e_1\|_\infty \|v_n - e_1\|_{L^2(M)} +$$

$$\|\varphi\|_\infty \|e_1 (d v_n - d e_1) + e_1 d e_1 - v_n d e_1\|_{L^2(M)} \Big]^2$$

$$\leq \left\| \frac{1}{e_1} \right\|_\infty^2 \Big[\|d\varphi\|_\infty \|e_1\|_\infty \|v_n - e_1\|_{L^2(M)} +$$

$$\|\varphi\|_\infty \|e_1\|_\infty \|d v_n - d e_1\|_{L^2(M)} + \|\varphi\|_\infty \|d e_1\|_\infty \|e_1 - v_n\|_{L^2(M)} \Big]^2;$$

so, since $\lim_{n\to+\infty} v_n = e_1$ for the norm $\|.\|_{H^1(M)}$ we have

$$\lim_{n\to+\infty} \int_M \left| d\left(\frac{\varphi v_n}{e_1}\right) - d\varphi \right|^2 d\mathcal{V}_g = 0.$$

Therefore, for all function $\varphi \in \mathcal{C}^\infty(M)$ we have $\lim_{n\to+\infty} \frac{\varphi v_n}{e_1} = \varphi$ for the norm $\|.\|_{H^1(M)}$.
Next, by density of $\mathcal{C}^\infty(M)$ in $H^1(M)$: for all function $f \in H^1(M)$ we have $\lim_{n\to+\infty} \frac{f v_n}{e_1} = f$. Since the sequence $\left(\frac{f v_n}{e_1}\right)_n \in \mathcal{H}_0^1(M-A)^{\mathbb{N}}$ we get finally that f belongs to space $H_0^1(M-A)$. Finally, it is easy to see that $(iv) \Rightarrow (iii)$. □

An obvious consequence of this lemma is the following result :

Proposition. *The spectrum of $-\Delta_g + V$ on the manifold (M, g) and on the manifold $(M-A, g)$ are equal if and only if $cap(A) = 0$.*

3.3 The Poincaré inequality

Now, let introduce the Poincaré inequality :

Theorem. *If $\lambda_1(M)$ denotes the first eigenvalue of the operator $-\Delta_g + V$ on the manifold (M, g), the following inequality*

$$\|u_A\|^2_{L^2(M)} \leq \frac{cap(A)}{\lambda_1(M)} \tag{3.2}$$

holds for all subset A of M.

Proof. The case $\mathrm{cap}(A) = 0$ is an obvious consequence of the lemma in section 3.2. Suppose here that $\mathrm{cap}(A) > 0$, then $\|u_A\|_{L^2(M)} > 0$. The first eigenvalue $\lambda_1(M)$ of the operator $-\Delta_g + V$ on the manifold (M, g) is given by :

$$\begin{aligned}
\lambda_1(M) &= \min_{\substack{E \subset H^1(M) \\ \dim(E)=1}} \max_{\substack{\varphi \in E \\ \varphi \neq 0}} \frac{\int_M |d\varphi|^2 + V |\varphi|^2 \, d\mathcal{V}_g}{\int_M |\varphi|^2 \, d\mathcal{V}_g} \\
&= \min_{\substack{\varphi \in H^1(M) \\ \varphi \neq 0}} \frac{\int_M |d\varphi|^2 + V |\varphi|^2 \, d\mathcal{V}_g}{\int_M |\varphi|^2 \, d\mathcal{V}_g}
\end{aligned}$$

Since u_A belongs to the space $H^1(M)$ we get $\lambda_1(M) \leq \frac{\mathrm{cap}(A)}{\|u_A\|^2_{L^2(M)}}$. □

3.4 The main theorem

Recall our main result :

Theorem. *Let (M, g) a compact Riemannian manifold. For all integer $k \geq 1$, there exists a constant C_k depending on the manifold of (M, g) and the potential V such that for all subset A of M we have :*

$$0 \leq \lambda_k(M - A) - \lambda_k(M) \leq C_k \sqrt{cap(A)}.$$

Remark. We can easily adapt the proof for a compact Riemannian manifold with boundary.

Proof. Let us denote by $(e_k)_{k \geq 1}$ an orthonormal basis of the space $L^2(M)$ with eigenfunctions of the operator $-\Delta_g + V$ on the manifold (M, g). For all integer $k \geq 1$, we consider the sets

$$F_k := \mathrm{span}\, \{e_1, e_2, \ldots, e_k\}$$

and

$$E_k := \left\{ f \left(1 - \frac{u_A}{e_1} \right), \, f \in F_k \right\}.$$

First, observe that $E_k \subset H^1_0(M - A)$. For all $j \in \{1, \ldots, k\}$ we introduce also the functions $\phi_j := e_j \left(1 - \frac{u_A}{e_1} \right) \in E_k$.

• **Step 1** : we compute the L^2-inner product $\langle \phi_i, \phi_j \rangle_{L^2(M)}$ for all pairs $(i,j) \in \{1,\ldots,k\}^2$:

$$\langle \phi_i, \phi_j \rangle_{L^2(M)} = \int_M e_i e_j \left(1 - \frac{u_A}{e_1}\right)^2 d\mathcal{V}_g$$

$$= \delta_{i,j} - 2\int_M \frac{e_i e_j}{e_1} u_A \, d\mathcal{V}_g + \int_M e_i e_j \frac{u_A^2}{e_1^2} \, d\mathcal{V}_g.$$

Thus, for all pair $(i,j) \in \{1,\ldots,k\}^2$ we get :

$$\left| \langle \phi_i, \phi_j \rangle_{L^2(M)} - \delta_{i,j} \right| \leq 2\int_M \left| \frac{e_i e_j}{e_1} u_A \right| \, d\mathcal{V}_g + \int_M \left| e_i e_j \frac{u_A^2}{e_1^2} \right| \, d\mathcal{V}_g,$$

hence, by Cauchy-Schwarz inequality we obtain

$$\left| \langle \phi_i, \phi_j \rangle_{L^2(M)} - \delta_{i,j} \right| \leq 2 \max_{1\leq i,j\leq k} \left\| \frac{e_i e_j}{e_1^2} \right\|_\infty \|u_A\|_{L^2(M)} + \max_{1\leq i,j\leq k} \left\| \frac{e_i e_j}{e_1^2} \right\|_\infty \|u_A\|^2_{L^2(M)}$$

$$\leq 2 \max_{1\leq i,j\leq k} \left\| \frac{e_i e_j}{e_1} \right\|_\infty \sqrt{\text{vol}(M)} \, \|u_A\|_{L^2(M)} + \max_{1\leq i,j\leq k} \left\| \frac{e_i e_j}{e_1^2} \right\|_\infty \|u_A\|^2_{L^2(M)}$$

hence by Poincaré inequality we have

$$\left| \langle \phi_i, \phi_j \rangle_{L^2(M)} - \delta_{i,j} \right| \leq B_{k,M} \left(\sqrt{\text{cap}(A)} + \text{cap}(A) \right)$$

where $B_k = B_k(e_1, e_2, ..., e_k, \lambda_1(M), M) \geq 0$, and since the eigenfunctions $e_1, e_2, ..., e_k$ and the eigenvalue $\lambda_1(M)$ depends only on (M,g) and V, for all integer k the constant B_k depends only on (M,g) and V, ie : $B_k = B_k(M,V)$.
Therefore, there exists $\varepsilon_k \in]0,1[$ (depends on the constant B_k) such that for all $A \subset M$ we have :

$$\text{cap}(A) \leq \varepsilon_k \Rightarrow \dim(E_k) = k \text{ and } \forall j \in \{1,...,k\}, \left| \|\phi_j\|^2_{L^2(M)} - 1 \right| \leq D_k \sqrt{\text{cap}(A)}$$

where (and for the same reasons as in the study of B_k) for all integer k, the constant D_k depends only on M and V, ie $D_k = D_k(M,V)$.

• **Step 2** : Let a function $\phi = f\left(1 - \frac{u_A}{e_1}\right) \in E_k$, with $f \in F_k$. Without loss generality we can assume that $\|f\|_{L^2(M)} = 1$, indeed : we have $R(\phi) = R\left(\frac{\phi}{\|f\|_{L^2(M)}}\right)$ and in our context we intereres in the Rayleigh quotient of ϕ (see the end of the final step of the proof).
Set $v_A := \frac{u_A}{e_1}$, we have :

$$\int_M |d\phi|^2 \, d\mathcal{V}_g = \int_M |df - d(fv_A)|^2 \, d\mathcal{V}_g$$

$$= \int_M |df|^2 \, d\mathcal{V}_g + \int_M |dfv_A + fdv_A|^2 \, d\mathcal{V}_g - 2\int_M df d(fv_A) \, d\mathcal{V}_g$$

$$= \int_M |df|^2 \, d\mathcal{V}_g + \int_M |dfv_A|^2 \, d\mathcal{V}_g + \int_M |fdv_A|^2 \, d\mathcal{V}_g$$

$$+2\int_M df dv_A f v_A \, d\mathcal{V}_g - 2\int_M |df|^2 v_A \, d\mathcal{V}_g - 2\int_M df dv_A f \, d\mathcal{V}_g$$

$$= \int_M |df|^2 \, d\mathcal{V}_g + \int_M |df v_A|^2 \, d\mathcal{V}_g + \int_M |f dv_A|^2 \, d\mathcal{V}_g$$

$$-2\int_M |df|^2 v_A \, d\mathcal{V}_g - 2\int_M df dv_A f \,(1 - v_A)\, d\mathcal{V}_g.$$

Recall we have $dv_A = \frac{du_A e_1 - u_A de_1}{e_1^2}$, and :

$$\int_M V |\phi|^2 \, d\mathcal{V}_g = \int_M V |f|^2 \, d\mathcal{V}_g - 2\int_M V |f|^2 v_A \, d\mathcal{V}_g + \int_M V |v_A f|^2 \, d\mathcal{V}_g$$

hence

$$\int_M |d\phi|^2 \, d\mathcal{V}_g + \int_M V |\phi|^2 \, d\mathcal{V}_g = \underbrace{\int_M |df|^2 \, d\mathcal{V}_g + \int_M V |f|^2 \, d\mathcal{V}_g}_{:=A(f)} + \underbrace{\int_M |df v_A|^2 \, d\mathcal{V}_g}_{:=B(f)}$$

$$+ \underbrace{\int_M |f dv_A|^2 \, d\mathcal{V}_g + \int_M V |v_A f|^2 \, d\mathcal{V}_g}_{:=C(f)} - 2\left(\underbrace{\int_M |df|^2 v_A \, d\mathcal{V}_g + \int_M V |f|^2 v_A \, d\mathcal{V}_g}_{:=D(f)} \right)$$

$$\underbrace{-2\int_M df dv_A f \,(1 - v_A)\, d\mathcal{V}_g}_{:=E(f)}.$$

♦ Study of $A(f) := \int_M |df|^2 \, d\mathcal{V}_g + \int_M V |f|^2 \, d\mathcal{V}_g \geq 0$: since $f \in F_k$ we can write $f = \sum_{i=1}^{k} \alpha_i e_i$ where $(\alpha_i)_{1 \leq i \leq k} \in \mathbb{R}^k$ and with $\sum_{i=1}^{k} \alpha_i^2 = 1$ (since $\|f\|_{L^2(M)} = 1$), thus we get

$$A(f) = \left\langle \sum_{j=1}^{k} \alpha_j de_j, \sum_{i=1}^{k} \alpha_i de_i \right\rangle_{L^2(M)} + \left\langle \sqrt{V} \sum_{j=1}^{k} \alpha_j e_j, \sqrt{V} \sum_{i=1}^{k} \alpha_i e_i \right\rangle_{L^2(M)}$$

$$= \sum_{i,j} \alpha_i \alpha_j \left(\langle de_j, de_i \rangle_{L^2(M)} + \int_M V e_j e_i \, d\mathcal{V}_g \right)$$

$$= \sum_{i,j} \alpha_i \alpha_j \left(- \langle e_j, \Delta_g e_i \rangle_{L^2(M)} + \int_M V e_j e_i \, d\mathcal{V}_g \right)$$

$$= \sum_{i,j} \alpha_i \alpha_j \langle e_j, (-\Delta_g + V)\, e_i \rangle_{L^2(M)}$$

$$= \sum_{i,j} \alpha_i \alpha_j \lambda_i(M) \langle e_j, e_i \rangle_{L^2(M)} = \sum_{i=1}^{k} \alpha_i^2 \lambda_i(M) \leq \lambda_k(M).$$

Hence, for all integer k, and for all function $f \in F_k$ such that $\|f\|_{L^2(M)} = 1$ we have

$$0 \le A(f) \le \lambda_k(M). \tag{3.3}$$

♦ Study of $B(f) := \int_M |d(f)v_A|^2 \, d\mathcal{V}_g$: here $v_A = \frac{u_A}{e_1}$ and $dv_A = \frac{du_A e_1 - u_A de_1}{e_1^2}$, so we get $B \le \|df\|_\infty^2 \|v_A\|_{L^2(M)}^2$ and, with the Poincaré inequality :

$$\|v_A\|_{L^2(M)}^2 \le \left\|\frac{1}{e_1}\right\|_\infty^2 \|u_A\|_{L^2(M)}^2 \le \left\|\frac{1}{e_1}\right\|_\infty^2 \frac{\text{cap}(A)}{\lambda_1(M)}$$

hence, for all integer k, and for all function $f \in F_k$ such that $\|f\|_{L^2(M)} = 1$ we have

$$0 \le B(f) \le E_k cap(A) \tag{3.4}$$

where $E_k = E_k(e_1, \lambda_1(M)) > 0$, moreover since the eigenfunction e_1 and the eigenvalue $\lambda_1(M)$ depends only on (M, g) and V, for all integer k the constant E_k depends only on (M, g) and V, ie : $E_k = E_k(M, V)$.

♦ Study of $C(f)$: here $C(f)$ is equal to $\underbrace{\int_M |f dv_A|^2 \, d\mathcal{V}_g}_{:=C_1(f)} + \underbrace{\int_M V |v_A f|^2 \, d\mathcal{V}_g}_{:=C_2(f)}$. Let us observe first $C_1(f)$:

$$C_1(f) \le \|f\|_\infty^2 \|dv_A\|_{L^2(M)}^2$$

and

$$\|dv_A\|_{L^2(M)}^2 = \int_M \left|\frac{du_A e_1 - u_A de_1}{e_1^2}\right|^2 d\mathcal{V}_g$$

$$\le \left\|\frac{1}{e_1}\right\|_\infty^2 \int_M |du_A e_1 - u_A de_1|^2 \, d\mathcal{V}_g$$

$$\le \left\|\frac{1}{e_1}\right\|_\infty^2 \left(\int_M |du_A e_1|^2 \, d\mathcal{V}_g + 2\int_M |du_A de_1 e_1 u_A| \, d\mathcal{V}_g + \int_M |de_1 u_A|^2 \, d\mathcal{V}_g\right)$$

$$\le \left\|\frac{1}{e_1}\right\|_\infty^2 \left(\|du_A\|_{L^2(M)}^2 \|e_1\|_\infty^2 + 2\|de_1\|_\infty \|e_1\|_\infty \|du_A\|_{L^2(M)} \|u_A\|_{L^2(M)} + \|de_1\|_\infty^2 \|u_A\|_{L^2(M)}^2\right).$$

Next we have also :

$$C_2(f) = \int_M V |v_A f|^2 \, d\mathcal{V}_g \le \|f\|_\infty^2 \int_M V |v_A|^2 \, d\mathcal{V}_g$$

$$\le \|f\|_\infty^2 \left\|\frac{1}{e_1}\right\|_\infty^2 \int_M V |u_A|^2 \, d\mathcal{V}_g.$$

Hence we get :

$$C(f) \le \|f\|_\infty^2 \left\|\frac{1}{e_1}\right\|_\infty^2 \Big[\|du_A\|_{L^2(M)}^2 \|e_1\|_\infty^2$$

$$+2\|de_1\|_\infty \|e_1\|_\infty \|du_A\|_{L^2(M)} \|u_A\|_{L^2(M)} + \|de_1\|_\infty^2 \|u_A\|_{L^2(M)}^2\Big]$$

$$+\|f\|_\infty^2 \left\|\frac{1}{e_1}\right\|_\infty^2 \int_M V\,|u_A|^2\, d\mathcal{V}_g$$

$$\leq \|f\|_\infty^2 \left\|\frac{1}{e_1}\right\|_\infty^2 \Big[\|du_A\|_{L^2(M)}^2 \|e_1\|_\infty^2 + 2\,\|de_1\|_\infty \|e_1\|_\infty \|du_A\|_{L^2(M)} \|u_A\|_{L^2(M)} + \|de_1\|_\infty^2 \|u_A\|_{L^2(M)}^2$$

$$+\int_M |du_A|^2\, d\mathcal{V}_g + \int_M V\,|u_A|^2\, d\mathcal{V}_g\Big]$$

$$\leq \|f\|_\infty^2 \left\|\frac{1}{e_1}\right\|_\infty^2 \Big[\|du_A\|_{L^2(M)}^2 + \|V\|_\infty \|u_A\|_{L^2(M)}^2$$

$$+2\,\|de_1\|_\infty \|e_1\|_\infty \|du_A\|_{L^2(M)} \|u_A\|_{L^2(M)} + \|de_1\|_\infty^2 \|u_A\|_{L^2(M)}^2\Big]\,;$$

so, since $\|du_A\|_{L^2(M)}^2 \leq \mathrm{cap}(A)$ and $\|u_A\|_{L^2(M)}^2 \leq \frac{\mathrm{cap}(A)}{\lambda_1(M)}$ we get for all integer k, and for all function $f \in F_k$ such that $\|f\|_{L^2(M)} = 1$:

$$0 \leq C(f) \leq F_k cap(A) \tag{3.5}$$

where $F_k = F_k\left(f, e_1, \lambda_1(M)\right) > 0$. Here, for k fixed, the constant F_k depends also on f, and f depends on the functions $f_1, f_2, \cdots, f_k$ (which are depends only on M and V) and on the scalars $\alpha_1, \alpha_2, \cdots, \alpha_k$; since $\sum_{i=1}^{k} \alpha_i^2 = 1$, all the $(\alpha_i)_{1\leq i\leq k}$ are bounded in $\mathbb{R}$, so finally, for all integer k the constant F_k can be bounded by a constant (we denotes also by $F_k = F_k(M, V)$) which depends only on M and V.

♦ Study of $|D(f)|$: we have

$$|D| = \left|\int_M |df|^2 v_A\, d\mathcal{V}_g + \int_M V\,|f|^2 v_A\, d\mathcal{V}_g\right|$$

$$\leq \|df\|_\infty^2 \left\|\frac{1}{e_1}\right\|_\infty \int_M \left|\frac{u_A}{e_1}\right| d\mathcal{V}_g + \left\|\frac{V\,|f|^2}{e_1}\right\|_\infty \int_M |u_A|\, d\mathcal{V}_g$$

$$\leq \max\left(\|df\|_\infty^2 \left\|\frac{1}{e_1}\right\|_\infty, \left\|\frac{V\,|f|^2}{e_1}\right\|_\infty\right) \int_M |u_A|\, d\mathcal{V}_g$$

$$\leq \max\left(\|df\|_\infty^2 \left\|\frac{1}{e_1}\right\|_\infty, \left\|\frac{V\,|f|^2}{e_1}\right\|_\infty\right) \sqrt{\mathrm{Vol}(M)}\, \|u_A\|_{L^2(M)}$$

$$\leq \max\left(\|df\|_\infty^2 \left\|\frac{1}{e_1}\right\|_\infty, \left\|\frac{V\,|f|^2}{e_1}\right\|_\infty\right) \sqrt{\mathrm{Vol}(M)}\sqrt{\frac{\mathrm{cap}(A)}{\lambda_1(M)}}.$$

Hence, for all integer k, and for all function $f \in F_k$ such that $\|f\|_{L^2(M)} = 1$:

$$|D(f)| \leq G_k \sqrt{cap(A)} \tag{3.6}$$

where (and for the same reasons as in the study of F, see the constant F_k) for all integer k, the constant G_k depends only on M and V, ie $G_k = G_k\,(M, V)$.

♦ Study of $|E(f)|$: recall that $E(f) = \int_M df dv_A f\,(1 - v_A)\, d\mathcal{V}_g$, hence

$$|E(f)| \leq \int_M |dfdv_A|\,|f|\;d\mathcal{V}_g + \int_M |dfdv_A|\,|fv_A|\;d\mathcal{V}_g.$$

For the first term $\int_M |dfdv_A|\,|f|\;d\mathcal{V}_g$ we have :

$$\int_M |dfdv_A|\,|f|\;d\mathcal{V}_g \leq \|f\|_\infty \|df\|_\infty \sqrt{\mathrm{Vol}(M)}\,\|dv_A\|_{L^2(M)}\,;$$

we have see in the study of $C(f)$ that

$$\|dv_A\|^2_{L^2}$$

$$\leq \left\|\frac{1}{e_1}\right\|^2_\infty \left(\|du_A\|^2_{L^2(M)}\|e_1\|^2_\infty + 2\|de_1\|_\infty\|e_1\|_\infty\|du_A\|_{L^2(M)}\|u_A\|_{L^2(M)} + \|de_1\|^2_\infty\|u_A\|^2_{L^2(M)}\right)$$

so with $K := \|f\|_\infty \|df\|_\infty \sqrt{\mathrm{Vol}(M)}\left\|\frac{1}{e_1}\right\|_\infty$ we get

$$\int_M |dfdv_A|\,|f|\;d\mathcal{V}_g$$

$$\leq K\sqrt{\|du_A\|^2_{L^2(M)}\|e_1\|^2_\infty + 2\|de_1\|_\infty\|e_1\|_\infty\|du_A\|_{L^2(M)}\|u_A\|_{L^2(M)} + \|de_1\|^2_\infty\|u_A\|^2_{L^2(M)}}$$

$$\leq K\sqrt{\mathrm{cap}(A)\,\|e_1\|^2_\infty + 2\|de_1\|_\infty\|e_1\|_\infty\sqrt{\mathrm{cap}(A)}\sqrt{\frac{\mathrm{cap}(A)}{\lambda_1(M)}} + \|de_1\|^2_\infty\frac{\mathrm{cap}(A)}{\lambda_1(M)}}$$

$$\leq H_k\sqrt{\mathrm{cap}(A)}$$

where (same reasons as above), for all integer k, the constant H_k depends only on M and V, ie $H_k = H_k(M,V)$.
Next, for the second term : $\int_M |dfdv_A|\,|fv_A|\;d\mathcal{V}_g$ we have :

$$\int_M |dfdv_A|\,|fv_A|\;d\mathcal{V}_g \leq \|df\|_\infty\|f\|_\infty\|dv_A\|_{L^2(M)}\|v_A\|_{L^2(M)}$$

$$\leq \|df\|_\infty\|f\|_\infty\|dv_A\|_{L^2(M)}\left\|\frac{1}{e_1}\right\|_\infty\|u_A\|_{L^2(M)}$$

$$\leq \|df\|_\infty\|f\|_\infty\left\|\frac{1}{e_1}\right\|_\infty\sqrt{\frac{\mathrm{cap}(A)}{\lambda_1(M)}}H_k\sqrt{\mathrm{cap}(A)}$$

$$\leq H'_{k,M}\mathrm{cap}(A).$$

where (same reasons as above), for all integer k, the constant H_k depends only on M and V, ie $H'_k = H_k(M,V)$.
So, for all integer k :

$$|E(f)| \leq H''_{k,M}\left(\sqrt{cap(A)} + cap(A)\right) \tag{3.7}$$

where $H''_k := H''_k(M,V)$.
Finally, with the study of $A(f), B(f), C(f), |D(f)|$ and $|E(f)|$, for all integer k,

for any function $\phi = f\left(1 - \frac{u_A}{e_1}\right) \in E_k$, with $f \in F_k$ such that $\|f\|_{L^2(M)} = 1$ we get :

$$\int_M |d\phi|^2 \, d\mathcal{V}_g + \int_M V\, |\phi|^2 \, d\mathcal{V}_g \leq \lambda_k(M) + I_k \left(\sqrt{cap(A)} + cap(A)\right) \tag{3.8}$$

where, for all integer k, the constant I_k depends only on M and V, ie : $I_k = I_k\,(M, V)$.

• **Step 3** : Now we claim that : for all $A \subset M$ such that $\text{cap}(A) \leq \varepsilon_k$ and for any function $\phi \in E_k$ we have :

$$\|\phi\|^2_{L^2(M)} \geq 1 - J'_{k,M}\sqrt{cap(A)} \tag{3.9}$$

where, for all integer k, the constant $J'_{k,M}$ depend only on M and V, ie : $J'_{k,M} = J'_{k,M}\,(M, V)$.
Indeed : let $\phi \in E_k$, we have seen below in step 1 that :

$$\text{cap}(A) \leq \varepsilon_k \Rightarrow \dim(E_k) = k \text{ and } \forall j \in \{1, ..., k\},\ \left|\|\phi_j\|^2_{L^2(M)} - 1\right| \leq D_k\sqrt{\text{cap}(A)}$$

therefore, since $\phi \in E_k$, we can write $\phi = (1 - v_A)f$ with $f = \sum_{i=1}^{k} \alpha_i e_i$ where $(\alpha_i)_{1 \leq i \leq k} \in \mathbb{R}^k$. As in the step two we can assume that $\|f\|_{L^2(M)} = 1$, hence we have $\sum_{i=1}^{k} \alpha_i^2 = 1$. Next, compute $\|\phi\|^2_{L^2(M)}$:

$$\|\phi\|^2_{L^2(M)} = \left\|\sum_{i=1}^{k} (1 - v_A)\, \alpha_i e_i\right\|^2_{L^2(M)} = \left\|\sum_{i=1}^{k} \alpha_i \phi_i\right\|^2_{L^2(M)}$$

$$= \sum_{i=1}^{k} \alpha_i^2\, \|\phi_i\|^2_{L^2(M)} + \sum_{i,j\ i \neq j} \alpha_i \alpha_j \left\langle \phi_i, \phi_j \right\rangle_{L^2(M)}.$$

And since

$$\sum_{i=1}^{k} \alpha_i^2\, \|\phi_i\|^2_{L^2(M)} = \sum_{i=1}^{k} \alpha_i^2 \left[1 - 2\int_M e_i^2 v_A \, d\mathcal{V}_g + \int_M e_i^2 v_A^2 \, d\mathcal{V}_g\right]$$

$$= 1 - \sum_{i=1}^{k} \alpha_i^2 \left[2\int_M e_i^2 v_A \, d\mathcal{V}_g - \int_M e_i^2 v_A^2 \, d\mathcal{V}_g\right]$$

$$= 1 - \sum_{i=1}^{k} \alpha_i^2 \int_M e_i^2 \left(2v_A - v_A^2\right) \, d\mathcal{V}_g;$$

hence

$$\|\phi\|^2_{L^2(M)} = 1 - \sum_{i=1}^{k} \alpha_i^2 \int_M e_i^2 \left(2v_A - v_A^2\right) \, d\mathcal{V}_g + \sum_{i,j\ i \neq j} \alpha_i \alpha_j \left\langle \phi_i, \phi_j \right\rangle_{L^2(M)}$$

we have seen in step 1 that, for $\mathrm{cap}(A)$ small enough :

$$\left|\left\langle \phi_i, \phi_j \right\rangle_{L^2(M)} - \delta_{i,j}\right| \leq B_k \left(\sqrt{\mathrm{cap}(A)} + \mathrm{cap}(A)\right)$$

hence, since all the $(\alpha_i)_{1\leq i\leq k}$ are bounded in $\mathbb{R}$, and for $\mathrm{cap}(A)$ small enough, we can find a constant $B'_{k,M}$ which depends only on M and V, ie $B'_k = B'_k\,(M,V)$ such that, for $\mathrm{cap}(A)$ small enough :

$$\left|\sum_{i,j\ i\neq j} \alpha_i\alpha_j \left\langle \phi_i, \phi_j \right\rangle_{L^2(M)}\right| \leq B'_k\sqrt{\mathrm{cap}(A)}$$

and finally, in the same spirit as in the estimations in section 2, there exists a constant $B''_{k,M}$ which depends only on M and V, ie $B''_k = B''_k\,(M,V)$ such that, for $\mathrm{cap}(A)$ small enough :

$$\left|\sum_{i=1}^{k} \alpha_i^2 \int_M e_i^2 \left(2v_A - v_A^2\right)\, d\mathcal{V}_g\right| \leq B''_k\sqrt{\mathrm{cap}(A)}$$

so finally we obtain :

$$\|\phi\|^2_{L^2(M)} \geq 1 - B'''_k\sqrt{\mathrm{cap}(A)}$$

where the constant B'''_k depend only on M and V, ie : $B'''_k := B'''_k\,(M,V)$.

• **Final step** : As a consequence from step 2 and 3, for all function $\phi \in E_k$ we get :

$$\frac{\int_M |d\phi|^2\, d\mathcal{V}_g + \int_M V\,|\phi|^2\, d\mathcal{V}_g}{\int_M \phi^2\, d\mathcal{V}_g} \leq \frac{\lambda_k(M) + I_k\left(\mathrm{cap}(A) + \sqrt{\mathrm{cap}(A)}\right)}{1 - B'''_k\sqrt{\mathrm{cap}(A)}}$$

hence for $\mathrm{cap}(A)$ small enough (ie : $\mathrm{cap}(A) \leq \varepsilon_k$) we have

$$\frac{\int_M |d\phi|^2\, d\mathcal{V}_g + \int_M V\,|\phi|^2\, d\mathcal{V}_g}{\int_M \phi^2\, d\mathcal{V}_g} \leq \lambda_k(M) + L_k\sqrt{\mathrm{cap}(A)}$$

where $L_k := L_k\,(M,V)$. Next, since for all $k \geq 1$

$$\lambda_k(M-A) = \min_{\substack{E\subset H_0^1(M-A)\\ \dim(E)=k}} \max_{\substack{\varphi\in E\\ \varphi\neq 0}} \frac{\int_M |d\varphi|^2\, d\mathcal{V}_g + \int_M V\,|\varphi|^2\, d\mathcal{V}_g}{\int_M \varphi^2\, d\mathcal{V}_g}$$

and since $\phi \in H_0^1(M-A)$, we get for all $k \geq 1$

$$\lambda_k(M-A) \leq \frac{\int_M |d\phi|^2\, d\mathcal{V}_g + \int_M V\,|\phi|^2\, d\mathcal{V}_g}{\int_M \phi^2\, d\mathcal{V}_g} \leq \lambda_k(M) + C_k\sqrt{\mathrm{cap}(A)}.$$

And the statement of the theorem is established. □

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