§ Weyl asymptotic formula [SpEx5.5.5WeylFormula]

Counting function $N(\lambda)=\#\{j:\lambda_j(M)\le\lambda\}$

Each eigenvalue is counted with its multiplicity。

For a square torus，each eigenvalue $\lambda_{m,n}=\frac{4\pi^2}{a^2}(m^2+n^2)$ corresponds to a point with integer coordinates (m,n) on the plane。

An approximate number of integer points inside the circle is given by the area of the circle。

$$N(\lambda)=\pi(\frac{a\sqrt{\lambda}}{2\pi})^2+R(\lambda)=\frac{a^2\lambda}{4\pi}+R(\lambda)\quad R(\lambda)=o(\lambda)\ \text{as}\ \lambda\to\infty$$

The problem of counting the number $N(\rho)$ of integer points inside a disk of radius ρ is called Gauss circle problem。

Conjecture (Hardy 1916)

For any $\varepsilon>0$，we have $R(\lambda)=O(\lambda^{1/2+\varepsilon})$ as $\lambda\to\infty$

§ Estimating the number of integer points in a disk。

1. Main Estimate $N(r)\approx\pi r^2$

   This is because, on average, each unit square on the integer grid contains about one point, and the area of the disk is πr^2。

2. Error Term E(r)

   The best known current bounds are of the form：$|E(r)|=O(r^\theta)$ with the smallest known value for θ currently around $\frac{517}{824}\approx 0.627$。

   It is conjectured that the optimal bound is $O(r^{1/2+\varepsilon})$ for any $\varepsilon>0$

r	$N(r)$	πr^2 (rounded)
0	1	0
1	5	3
2	13	13
3	29	28
4	49	50
5	81	79

G.H.Hardy　S.Ramanujan

§ Bessel function and lattice point counting

Bessel functions solve differential equations that arise naturally when transforming radially symmetric functions (like the disk's characteristic function) in two dimensions。

The appearance of Bessel functions is tied to the circular symmetry of the problem。