§ Virabting Drum [Spec005-3WaveEquation]

$U(t,x,y): R_+ \times \Omega \to R$

$U_{tt} - a^2 \Delta U = 0 \quad U|_{\partial\Omega} = 0$

Let u(t,x,y)=T(t)u(x,y)

$$\begin{cases} -\Delta u = \lambda u \\ u|_{\partial\Omega} = 0 \end{cases}$$

The Dirichlet problem usually can not be explicitly solved。

However，for certain geometries — for example，for a rectangle or for a disk — that could be done by using once again the separation of variables。

Let $R_{a,b} = (0,a) \times (0,b)$ be a rectangle with sides a and b. Show that

$$\lambda^{\mathrm{D}}_{k,m} = \pi^2\left(\frac{k^2}{a^2} + \frac{m^2}{b^2}\right), \qquad k,m = 1,2,\ldots, \tag{1.1.11}$$

are the eigenvalues of the Dirichlet problem (1.1.9)–(1.1.10) on $R_{a,b}$, and the corresponding eigenfunctions are given by

$$u^{\mathrm{D}}_{k,m}(x,y) = \sin\frac{k\pi}{a}x \sin\frac{m\pi}{b}y. \tag{1.1.12}$$

Prove that these functions form an orthogonal basis in $L^2(R_{a,b})$.

§ Problem for a disk [PDE701Harmonic]

$-\Delta u = \lambda u$ subject to the Dirichlet condition $u|_{\partial\Omega} = 0$ or Neumann condition

$\frac{\partial u}{\partial r}|_{r=1} = 0$

Switch to polar coordinates (r,φ)

$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}$ for the Laplacian in planar polar coordinates，and looking the solutions in the form $u(r,\varphi) = \sum_{n=-\infty}^{\infty} u_n(r) e^{in\varphi}$

Solution Summary：

1. Dirichlet problem

   $\lambda_{m,n} = (j_{m,n})^2$，where $j_{m,n}$ is the n-th positive zero of the Bessel function $J_m(x)$

   for m=0,1,2,3,...and n=1,2,...

- Eigenfunctions:
  - For $m = 0$: $u_{0,n}(r,\theta) = J_0(j_{0,n} r)$
  - For $m \geq 1$:
    $u_{m,n}^{(1)}(r,\theta) = J_m(j_{m,n} r)\cos(m\theta)$,
    $u_{m,n}^{(2)}(r,\theta) = J_m(j_{m,n} r)\sin(m\theta)$.
    Each eigenvalue $\lambda_{m,n}^{(D)}$ has multiplicity 1 if $m = 0$ and multiplicity 2 if $m \geq 1$.

2. Neumann problem

- Eigenvalues:
  - $\lambda_{0,0}^{(N)} = 0$ (multiplicity 1),
  - $\lambda_{0,n}^{(N)} = (j_{1,n})^2$ for $n = 1, 2, 3, \ldots$,
  - $\lambda_{m,n}^{(N)} = (j'_{m,n})^2$ for $m = 1, 2, \ldots$ and $n = 1, 2, 3, \ldots$.
    Here, $j_{1,n}$ is the n-th positive zero of $J_1(x)$, and $j'_{m,n}$ is the n-th positive zero of $\frac{d}{dx} J_m(x)$

- Eigenfunctions:
  - For $\lambda = 0$: $u_{0,0}(r,\theta) = 1$ (constant function),
  - For $m = 0, n \geq 1$: $u_{0,n}(r,\theta) = J_0(j_{1,n} r)$,
  - For $m \geq 1, n \geq 1$:
    $u_{m,n}^{(1)}(r,\theta) = J_m(j'_{m,n} r)\cos(m\theta)$,
    $u_{m,n}^{(2)}(r,\theta) = J_m(j'_{m,n} r)\sin(m\theta)$.
    Eigenvalues $\lambda_{0,n}^{(N)}$ $(n \geq 1)$ have multiplicity 1, and $\lambda_{m,n}^{(N)}$ $(m \geq 1)$ have multiplicity 2.

Let us describe the eigenvalues and eigenfunctions of the Dirichlet and Neumann problems in the unit disk $\mathbb{D}$. Switching to polar coordinates (r, φ), using the standard expression

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}$$

for the Laplacian in planar polar coordinates, and looking for solutions of (1.1.9) in the form

$$u(r,\varphi) = \sum_{m=-\infty}^{+\infty} u_m(r) e^{im\varphi},$$

we arrive at the equations

$$u_m''(r) + \frac{1}{r} u_m'(r) + \left(\lambda - \frac{m^2}{r^2}\right) u_m(r) = 0 \tag{1.1.15}$$

for unknown functions u_m.

The equations (1.1.15) are closely related to the *Bessel equation*

$$y''(r) + \frac{1}{r} y'(r) + \left(1 - \frac{m^2}{r^2}\right) y(r) = 0. \tag{1.1.16}$$

This solution fully characterizes the eigenvalues and eigenfunctions for both boundary conditions in the unit disk。

The eigenfunctions form orthogonal bases for L^2 spaces over the disk under respective boundary conditions。

The Bessel differential equation：

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-\nu^2)y=0$$

General solution is $y(x)=c_1J_\nu(x)+c_2Y_\nu(x)$

The modified Bessel differential equation：

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-(x^2+\nu^2)y=0$$

$$y(x)=c_1I_\nu(x)+c_2K_\nu(x)$$

$$I_\nu(x)=\sum_{k=0}^{\infty}\frac{1}{k!\Gamma(\nu+k+1)}(\frac{x}{2})^{\nu+2k} \qquad K_\nu(x)=\frac{\pi}{2}\frac{I_{-\nu}(x)-I_\nu(x)}{\sin\nu x}$$