§ Vibrating String [PDE102WaveEqyation]

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \text{，} u(0,t) = u(l,t) = 0$$

設 u(x,t)=X(x)T(t) 則 $u_{tt} = X(x)T''(t), u_{xx} = X''(x)T(t)$

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{c^2 T(t)} = -\lambda$$

$-X''(x) = \lambda X(x)$ …Sturm-Liouville eigenvalue problem

$X(0) = X(l) = 0$

1. $\lambda_n = (\frac{\pi n}{l})^2, X_n = \sin(\frac{\pi n}{l} x), m = 1, 2, 3, \ldots$

$X(x) = \sin\sqrt{\lambda}x$ ， $X(l) = \sin\sqrt{\lambda}l = 0 \Rightarrow \sqrt{\lambda}l = n\pi \Rightarrow \lambda = (\frac{n\pi}{l})^2$

$\therefore \lambda_n = (\frac{\pi n}{l})^2, X_n = \sin(\frac{\pi n}{l} x), m = 1, 2, 3, \ldots$

2. $\int_0^l X_m(x) X_n(x) dx = 0$ for $m \neq n$

$\sin x \sin y = \cos(x-y) - \cos(x+y)$

$\sin(my)\sin(ny) = \cos[(m-n)y] - \cos[(m+n)y]$

$$\int_0^l \sin\frac{m\pi x}{l} \sin\frac{n\pi x}{l} dx \overset{y=\pi x/l}{=} \frac{l}{\pi}\int_0^{\pi} \sin(my)\sin(ny) dy = 0$$

同理 $-T''(t) = c^2 T(t)$

$$T_n(t) = A_n \cos(\frac{c\pi n}{l} t) + B_n \sin(\frac{c\pi n}{l} t)$$

3. Show that the constants A_m and B_m, $m \in \mathbb{N}$, are uniquely determined by the initial conditions $u(0,x) = \varphi(x)$ (initial position), $u_t(0,x) = \psi(x)$ (initial velocity). Calculate A_m and B_m using the Fourier decompositions of the functions φ and ψ.

Fourier 展開解的結構 $u(x,t) = \sum_{n=1}^{\infty} [A_n \cos(\frac{nc\pi t}{l}) + B_n \sin(\frac{nc\pi t}{l})] \sin(\frac{n\pi x}{l})$

$$u(x,0) = \varphi(x) = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi x}{l})$$

$$A_n = \frac{2}{l}\int_0^l \varphi(x) \sin(\frac{n\pi x}{l}) dx$$

$$u_t(x,0) = \psi(x) = \sum_{n=1}^{\infty} \frac{n\pi x}{l} B_n \sin(\frac{n\pi x}{l})$$

$$B_n = \frac{2}{n\pi x}\int_0^l \psi(x) \sin(\frac{n\pi x}{l}) dx$$

The natural frequencies are $\omega_n = c\sqrt{\lambda_n} = \dfrac{c\pi n}{l}, n \in N$

$\omega_1 = \dfrac{c\pi}{l}$ ；principal frequency (fundamental tone 基頻)

overtones(泛音)； $\omega_n = n\omega_1, n \geq 2$

4. For Neumann boundary condition $u_x(t,0) = u_x(t,l) = 0$ 比較(1)Dirichlet (2)Neumann condition 的 eigenfrequencies

特性	Dirichlet 條件	Neumann 條件
端點約束	固定 $(u = 0)$	自由 $(\partial u/\partial x = 0)$
特徵值 λ_n	$\left(\frac{n\pi}{l}\right)^2$ $(n \geq 1)$	$\left(\frac{n\pi}{l}\right)^2$ $(n \geq 0)$
空間特徵函數	$\sin\left(\frac{n\pi x}{l}\right)$	$\cos\left(\frac{n\pi x}{l}\right)$
基頻 $(n = 1)$	$\omega_1 = \frac{c\pi}{l}$	$\omega_1 = \frac{c\pi}{l}$
泛音 $(n \geq 2)$	$n\omega_1$	$n\omega_1$
是否存在零頻 $(n = 0)$	否	是（靜態解）

(1) Dirichlet 邊界條件(固定端點)：

空間方程解

解得特徵值與特徵函數：

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, \quad X_n(x) = \sin\left(\frac{n\pi x}{l}\right), \quad n = 1, 2, 3, \ldots$$

時間方程解

$$\omega_n = c\sqrt{\lambda_n} = \frac{c\pi n}{l}, n \in N$$

(2) Neumann 邊界條件(自由端點)：

空間方程解

解得特徵值與特徵函數：

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, \quad X_n(x) = \cos\left(\frac{n\pi x}{l}\right), \quad n = 0, 1, 2, \ldots$$

其中 $n = 0$ 對應常數解 $\lambda_0 = 0$。

時間方程解

固有頻率 $\omega_n = c\sqrt{\lambda_n} = \frac{cn\pi}{l}, n = 1, 2, 3, \ldots$

基頻 $\omega_1 = \frac{c\pi}{l}$ 泛音 $\omega_n = n\omega_1, n \geq 2$

結論：頻譜相同
本質區別

Dirichlet 條件：無零頻解，所有模態均為振動。

Neumann 條件：存在零頻解（n=0），允許系統整體平移或靜態變形。

§ S^1 上的一維振動 [lesson001]

特徵值問題：$-\frac{d^2u}{dx^2} = \lambda u, u(x+2\pi) = u(x), \lambda \geq 0$

$\lambda_0 = 0$，特徵函數：{1}

$\lambda_n = n^2$，特徵函數：$\{\cos(nx), \sin(nx)\}$ (與 $e^{in\theta}$ 等價，構成 $L^2(S^1)$ 的一組完備正交基。

譜的完整描述：
Laplace 算子 Δ 的譜是離散的，譜 $(\Delta) = \{0, 1^2, 2^2, \ldots, n^2, \ldots\} = \{0, 1, 4, 9, \ldots, n^2, \ldots\}$

特徵值 $\lambda_n = n^2$ 的重數(multiplicity)：

$\lambda_0 = 0$ ：重數 1(常數函數)

$\lambda_n = n^2 (n \geq 1)$ ：重數 2 (cos(nx)與 sin(nx)線性獨立)