§ Spectrum of the Laplacian on a compact manifold

4.1 Physical examples

4.1.1 The wave equation on a string

One-dimensional string of length L>0

Wave equation $\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}=0$ with u(0,t)=u(L,t)=0 and $u(x,0)=u_0(x)$

The family $\{e_n(x):=\sin(\frac{n\pi x}{L})\}_{n\geq 1}$ form a Hilbert basis of

$H:=\{f\in L^2([0,T]); f(0)=f(T)=0\}$ ，and for any ineger $n\geq 1$ we have the relation

$$-\frac{\partial^2 e_n}{\partial x^2}(x)=\frac{n^2\pi^2}{L^2}e_n(x)$$

也就是說，算子$-\frac{\partial^2}{\partial x^2}$(with the boundary codition u(0,t)=u(L,t)=0)的譜為$\{\frac{n^2\pi^2}{L^2}\}_{n\geq 1}$

所以若 u(x,t)是 wave equation 的解，則存在一個實數序列$\{a_n\}$使得

$u(x,t)=\sum_{n=1}^{\infty}a_n(t)e_n(x)$ 代入 wave equation 得

$\sum_{n=1}^{\infty}(a_n''(t)+\frac{n^2\pi^2}{L^2}a_n(t))e_n(x)=0$ ，我們得到一個 ODE $a_n''(t)+\frac{n^2\pi^2}{L^2}a_n(t)=0$

因此$a_n(t)=A_n\cos(\frac{n\pi t}{L})+B_n\sin(\frac{n\pi t}{L})\quad n\geq 1\in Z, t\geq 0$

$$u(x,t)=\sum_{n=1}^{\infty}(A_n\cos(\frac{n\pi t}{L})+B_n\sin(\frac{n\pi t}{L}))\sin(\frac{n\pi x}{L})$$

4.1.2 The heat equation

4.1.3 The Schrodinger equation