§ Eigenvalues of Laplace-Beltrami operator on S^2

Using spherical coordinates (θ,ϕ)，the Laplace-Beltrami operator on S^2 is：

$$\Delta_{S^2} = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta}) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}...(*)$$

We seek eigenfunctions $Y(\theta,\phi)$，and eigenvalues λ such that $\Delta_{S^2}Y = -\lambda Y$

Separation of variables

設$Y(\theta,\phi)=\Theta(\theta)\Phi(\phi)$代入(*)得

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial\Theta}{\partial\theta})\Phi + \frac{1}{\sin^2\theta}\Theta\frac{\partial^2\Phi}{\partial\phi^2} = -\lambda\Theta\Phi$$

，同除以$\Theta\Phi$，再同乘以$\sin^2\theta$

$$\frac{\sin\theta}{\Theta}\frac{d}{d\theta}(\sin\theta\frac{d\Theta}{d\theta}) + \lambda\sin^2\theta + \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = 0$$

設$\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = -m^2$，則

$$\begin{cases} \dfrac{d^2\Phi}{d\phi^2} + m^2\Phi = 0...(1) \\ \dfrac{\sin\theta}{\Theta}\dfrac{d}{d\theta}(\sin\theta\dfrac{d\Theta}{d\theta}) + (\lambda\sin^2\theta - m^2)\Theta = 0...(2) \end{cases}$$

(1) 式是方位角方程式，(2)式是極角方程式。

由(1) $\Phi(\phi)=e^{im\phi}$，其中 m 是整數(因為 BC $\Phi(\phi+2\pi)=\Phi$)

(2) 的部分，令$x=\cos\theta$，$\sin\theta=\sqrt{1-x^2}$，$\frac{d}{d\theta} = -\sin\theta\frac{d}{dx} = -\sqrt{1-x^2}\frac{d}{dx}\cdots(3)$

設$P(x)=\Theta(\theta)$ 代入(3) $\frac{d\Theta}{d\theta} = -\sqrt{1-x^2}\frac{dP}{dx}$，$\sin\theta\frac{d\Theta}{d\theta} = -(1-x^2)\frac{dP}{dx}$

計算$\frac{d}{d\theta}(\sin\theta\frac{d\Theta}{d\theta}) = \frac{d}{d\theta}(-(1-x^2)\frac{dP}{dx}) = \sqrt{1-x^2}\frac{d}{dx}((1-x^2)\frac{dP}{dx})$

$$\sin\theta\frac{d}{d\theta}(\sin\theta\frac{d\Theta}{d\theta}) = (1-x^2)\frac{d}{dx}((1-x^2)\frac{dP}{dx})$$

(2)式變成$(1-x^2)\frac{d}{dx}((1-x^2)\frac{dP}{dx}) + (\lambda(1-x^2)-m^2)P = 0$同處以$(1-x^2)$後展開得

$$(1-x^2)\frac{d^2P}{dx^2} - 2x\frac{dP}{dx} + (\lambda - \frac{m^2}{1-x^2})P = 0$$

得到 associated Legendre equation：

$$\frac{d}{dx}((1-x^2)\frac{dP}{dx}) + (\lambda - \frac{m^2}{1-x^2})P = 0$$

Eigenvalue $\lambda = l(l+1)$由 associated Legendre equation 解的存在性決定。

最後得到 eigenfunctions $Y_l^m(\theta,\phi) \propto P_l^m(\cos\theta)e^{im\phi}, \lambda_l = l(l+1)$

結論：

在 S^2 上的 Laplace-Beltrami operator 有一組完備的 eigenfunction…spherical harmonic function $Y_l^m(\theta,\varphi)$，滿足 $\Delta_{S^2} Y_l^m(\theta,\varphi) = -l(l+1)Y_l^m(\theta,\varphi)$

$l = 0,1,2,3,..,m = -l,-l+1,...,l-1,l$

後記

1. $\Delta := \frac{1}{\sqrt{g}}\partial_i(\sqrt{g}g^{ij}\partial_j)$，$S^2 \subset R^3$，in spherical coordinates (θ,ϕ)

   $ds^2 = d\theta^2 + \sin^2\theta d\varphi^2$，$\sqrt{g} = \sin\theta$ then $\Delta_{S^2} = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta}) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$

2. 完整的解為 $Y_l^m(\theta,\phi) = \sqrt{\frac{2l+(l-m)!}{4\pi(1+m)!}}P_l^m(\cos\theta)e^{im\phi}$

3. Legendre polynomial：

The Legendre polynomials satisfy the **Legendre differential equation**:

$$(1-x^2)\frac{d^2P_n(x)}{dx^2} - 2x\frac{dP_n(x)}{dx} + n(n+1)P_n(x) = 0,$$

where n is a non-negative integer (the polynomial degree).

2. **Orthogonality:**
   They are orthogonal on the interval $[-1,1]$ with respect to the weight function $w(x) = 1$:

$$\int_{-1}^{1} P_m(x)P_n(x)\,dx = 0 \quad \text{if } m \neq n.$$

   For $m = n$, the integral evaluates to $\frac{2}{2n+1}$.

3. **Rodrigues' Formula:**
   The n-th Legendre polynomial can be generated using:

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left((x^2-1)^n\right).$$

First few Legendre polynomials：

$$P_0(x) = 1, P_1(x) = x, P_2(x) = \frac{1}{2}(3x^2-1), P_3(x) = \frac{1}{2}(5x^3-3x), P_4(x) = \frac{1}{8}(35x^4-30x^2+3)$$

4. 經典物理中，重力與靜電場滿足 $\Delta\Phi = 0$，當問題約束在球面上時就變成 $\Delta_{S^2}\Phi = 0$；例如球面上的熱擴散方程 $\frac{\partial u}{\partial t} = \Delta_{S^2} u$

5. 陀螺運動方程的
   (1)對稱性 (2)角動量量子化 (3)球面幾何與 Laplace-Beltrami operator 有深層的關係。