

§ The Laplacian on a compact Riemannian manifold

3.1 Basic Riemannian Geometry

A connection on M is a map $D: TM \times \Gamma(M) \rightarrow TM$

$$D_X Y := D(X, Y)$$

Lie derivative $L_X(\alpha) := \lim_{t \rightarrow 0} \frac{\phi_t^* \alpha - \alpha}{t}$

Cartan formula $L_X(\alpha) = \iota_X d\alpha + d(\iota_X \alpha)$ for all $\alpha \in \Omega^k(M)$

In particular, for a function f , we get $L_X f = \iota_X df = df(X)$

The Levi-Civita connection is the unique connection on TM such that

(1) D is torsion free

(2) D is compatible with the metric g

For a Riemannian manifold (M, g) of dimension n and for a local chart $\phi: U \subset M \rightarrow \mathbb{R}^n$ which coordinates are denoted by (x^1, x^2, \dots, x^n) , we denote by $\left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \dots, \frac{\partial}{\partial x^n}\right)$ the associated vector fields. Then we have the following local expression of the Levi-Civita connection:

$$D_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \sum_{k=1}^n \Gamma_{ij}^k \frac{\partial}{\partial x^k},$$

where Γ_{ij}^k denote the *Christoffel symbols*.

The geodesic $\nabla_T T = 0$

The exponential map

Theorem 3.1.19 (Hopf–Rinow Theorem). *Let (M, g) be a Riemannian connected manifold. The following conditions are equivalent:*

- (i) *The manifold (M, g) is complete.*
- (ii) *There exists $x \in M$ such that the map \exp_x is defined on the whole tangent space $T_x M$.*
- (iii) *The map \exp_x is defined on the whole tangent space $T_x M$ for all $x \in M$.*
- (iv) *Compact sets of M are exactly closed and bounded sets of M .*

Moreover, any of these conditions implies that every pair of points $(x, y) \in M^2$ can be joined by a geodesic curve.

From now on, all Riemannian manifolds are supposed to be complete.

The curvature tensor

$$R(X, Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X, Y]}Z$$

For a fixed point x on the manifold M , the *sectional curvature* of a 2-plane $P \subset T_x M$ spanned by a basis X_1, X_2 is the number

$$K_x(P) := \frac{g(R(X_1, X_2)X_2, X_1)}{g(X_1 \wedge X_2, X_1 \wedge X_2)} = \frac{R(X_1, X_2, X_1, X_2)}{\|X_1 \wedge X_2\|^2},$$

The Ricci curvature tensor

$$Ric_x(X, X) := \sum_{i=1}^n R(X, e_i, X, e_i) \quad \text{for any point } x \in M \text{ and any vector } X \in T_x M,$$

where $\{e_i\}$ is an orthonormal basis of the vector space $T_x M$.

	Ricci curvature	Scalar curvature
R^n	0	0
S^n	$(n-1)g$	$n(n-1)$
H^n	$-(n-1)g$	$-n(n-1)$

§ divergence operator

$$div(X) := tr(Y \rightarrow \nabla_Y X)$$

For a local chart $\phi: U \subset M \rightarrow \mathbb{R}^n$ of M and for a vector field $X = \sum_j X^j \frac{\partial}{\partial x^j}$

$$div(X) = \frac{1}{\sqrt{g}} \sum_{i=1}^n \frac{\partial(\sqrt{g} X^i)}{\partial x^i}$$