§ Eigenvalues

§ 01 The wave equation on a string

$u_{tt} - u_{xx} = 0$ ⋯(*) u(0,t)=u(L,t)=0，u(x,0)=$u_0(x)$

$e_n(x) := \sin(\frac{n\pi x}{L}), n \ge 1$

$H := \{ f \in L^2([0,T]) \mid f(0) = f(T) = 0\}$ 則 $e_n(x)$ 形成 H 的 basis。

We have the relation $-\frac{\partial^2 e_n(x)}{\partial^2 x}(x) = \frac{n^2\pi^2}{L^2} e_n(x)$ 是 operator $-\frac{\partial^2}{\partial x^2}$ 的譜(spectrum，包含 eigenvalues and eigenfunctions)

$u(x,t) = \sum_{n=1}^{\infty} a_n(t) e_n(x)$ 代入(*) 得

$\sum_{n=1}^{\infty}(a_n^{''}(t) + \frac{n^2\pi^2}{L^2} a_n(t)) e_n(x) = 0 \quad a_n(t) = A_n \cos(\frac{n\pi t}{L}) + B_n \sin(\frac{n\pi t}{L})$

$u(x,t) = \sum_{n=1}^{\infty} [\mathrm{A}_n \cos(\frac{n\pi t}{L}) + B \sin(\frac{n\pi t}{L})] \sin(\frac{n\pi x}{L})$

Spectrum of the operator $-\frac{\partial^2}{\partial x^2}$ is $\left\{\frac{n^2\pi^2}{L^2}\right\}_{n\ge 1}$

注意到當 $L = 2\pi$，則譜變成 $\{n^2\}_{n\ge 1}$，例如定義的空間為單位圓。

§ 02 The heat equation

$\Omega \subset R^3 \quad u_t - \Delta u = 0$ with BC ⋯ and initial condition $u(x,0) = u_0(x)$

Separation of variables $u(x,t) = \varphi(t)\omega(x)$ then $\frac{\varphi'(t)}{\varphi(t)} = \frac{\Delta\omega(x)}{\omega(x)} = -\lambda$

$\varphi(t) = \varphi(0) e^{-\lambda t}$

$-\Delta\omega(x) = \lambda\omega(x)$，with $\omega(x) = 0$ for $x \in \partial\Omega$ then λ is the eigenvalue of $-\Delta$ (with Dirichlet condition)

$(e_k)_{k\ge 0}$ 形成一組 basis，$-\Delta e_k = \lambda_k e_k$

$u(x,t) = \sum_{k=1}^{\infty} c_k e^{-\lambda_k t} e_k(x)$

若 $\Omega = [0,a]\times[0,b]\times[0,c]$ 則 $e_k(x,y,z) = \sin(\frac{k_x \pi x}{a}) \sin(\frac{k_y \pi y}{b}) \sin(\frac{k_c \pi z}{c})$

對應的特徵值為 $\lambda_k = \pi^2(\frac{k_x^2}{a^2} + \frac{k_y^2}{b^2} + \frac{k_z^2}{c^2})$

若 Ω 是球形區域，則涉及球諧函數與球貝塞爾函數⋯