§ Vibrating Drum [Spec005-3WaveEquation]

$U(t,x,y): R_+ \times \Omega \to R$

$U_{tt} - a^2 \Delta U = 0$ ， $U|_{\partial\Omega} = 0$

Let U(t,x,y)=T(t)u(x,y)

$$\begin{cases} -\Delta u = \lambda u \\ u|_{\partial\Omega} = 0 \end{cases}$$

The Dirichlet problem usually can not be explicitly solved。
However，for certain geometries — for example，for a rectangle or for a disk — that could be done by using once again the separation of variables。

§ Problem for a disk [PDE701Harmonic]

$-\Delta u = \lambda u$ subject to the Dirichlet condition $u|_{\partial\Omega} = 0$ or Neumann condition

$\frac{\partial u}{\partial r}|_{r=1} = 0$

Switch to polar coordinates (r,φ)

$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}$ for the Laplacian in planar polar coordinates，and looking the solutions in the form $u(r,\varphi) = \sum_{n=-\infty}^{\infty} u_n(r) e^{in\varphi}$

Solution Summary：

1. Dirichlet problem

   $\lambda_{m,n} = (j_{m,n})^2$ ，where $j_{m,n}$ is the n-th positive zero of the Bessel function $J_m(x)$

   for m=0,1,2,3,...and n=1,2,...

   - **Eigenfunctions:**
     - For $m = 0$: $u_{0,n}(r,\theta) = J_0(j_{0,n} r)$
     - For $m \geq 1$:
       $u^{(1)}_{m,n}(r,\theta) = J_m(j_{m,n} r)\cos(m\theta)$,
       $u^{(2)}_{m,n}(r,\theta) = J_m(j_{m,n} r)\sin(m\theta)$.
       Each eigenvalue $\lambda^{(D)}_{m,n}$ has multiplicity 1 if $m = 0$ and multiplicity 2 if $m \geq 1$.

2. Neumann problem

- Eigenvalues:
  - $\lambda_{0,0}^{(N)} = 0$ (multiplicity 1),
  - $\lambda_{0,n}^{(N)} = (j_{1,n})^2$ for $n = 1, 2, 3, \ldots$,
  - $\lambda_{m,n}^{(N)} = (j'_{m,n})^2$ for $m = 1, 2, \ldots$ and $n = 1, 2, 3, \ldots$.

    Here, $j_{1,n}$ is the n-th positive zero of $J_1(x)$, and $j'_{m,n}$ is the n-th positive zero of $\frac{d}{dx} J_m(x)$

- Eigenfunctions:
  - For $\lambda = 0$: $u_{0,0}(r, \theta) = 1$ (constant function),
  - For $m = 0, n \geq 1$: $u_{0,n}(r, \theta) = J_0(j_{1,n} r)$,
  - For $m \geq 1, n \geq 1$:

    $u_{m,n}^{(1)}(r, \theta) = J_m(j'_{m,n} r) \cos(m\theta)$,

    $u_{m,n}^{(2)}(r, \theta) = J_m(j'_{m,n} r) \sin(m\theta)$.

    Eigenvalues $\lambda_{0,n}^{(N)}$ $(n \geq 1)$ have multiplicity 1, and $\lambda_{m,n}^{(N)}$ $(m \geq 1)$ have multiplicity 2.

Let us describe the eigenvalues and eigenfunctions of the Dirichlet and Neumann problems in the unit disk $\mathbb{D}$. Switching to polar coordinates (r, φ), using the standard expression

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}$$

for the Laplacian in planar polar coordinates, and looking for solutions of (1.1.9) in the form

$$u(r, \varphi) = \sum_{m=-\infty}^{+\infty} u_m(r) e^{im\varphi},$$

we arrive at the equations

$$u''_m(r) + \frac{1}{r} u'_m(r) + \left(\lambda - \frac{m^2}{r^2}\right) u_m(r) = 0 \qquad (1.1.15)$$

for unknown functions u_m.

The equations (1.1.15) are closely related to the *Bessel equation*

$$y''(r) + \frac{1}{r} y'(r) + \left(1 - \frac{m^2}{r^2}\right) y(r) = 0. \qquad (1.1.16)$$

This solution fully characterizes the eigenvalues and eigenfunctions for both boundary conditions in the unit disk。

The eigenfunctions form orthogonal bases for L^2 spaces over the disk under respective boundary conditions。

The Bessel differential equation：

$x^2 \dfrac{d^2 y}{dx^2} + x \dfrac{dy}{dx} + (x^2 - \nu^2) y = 0$ General solution is $y(x) = c_1 J_\nu(x) + c_2 Y_\nu(x)$

The modified Bessel differential equation：

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-(x^2+\nu^2)y=0$$

$$y(x)=c_1I_\nu(x)+c_2K_\nu(x)$$

$$I_\nu(x)=\sum_{k=0}^{\infty}\frac{1}{k!\Gamma(\nu+k+1)}(\frac{x}{2})^{\nu+2k}$$ ， $$K_\nu(x)=\frac{\pi}{2}\frac{I_{-\nu}(x)\text{-}I_\nu(x)}{\sin\nu x}$$