

§ Vibrating String [PDE102WaveEquation]

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad u(0, t) = u(l, t) = 0$$

設 $u(x, t) = X(x)T(t)$ 則 $u_{tt} = X(x)T''(t), u_{xx} = X''(x)T(t)$

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{c^2 T(t)} = -\lambda$$

$-X''(x) = \lambda X(x)$... Sturm-Liouville eigenvalue problem

$$X(0) = X(l) = 0$$

$$1. \quad \lambda_n = \left(\frac{\pi n}{l}\right)^2, X_n = \sin\left(\frac{\pi n}{l}x\right), n = 1, 2, 3, \dots$$

$$X(x) = \sin\sqrt{\lambda}x, \quad X(l) = \sin\sqrt{\lambda}l = 0 \Rightarrow \sqrt{\lambda}l = n\pi \Rightarrow \lambda = \left(\frac{n\pi}{l}\right)^2$$

$$\therefore \lambda_n = \left(\frac{\pi n}{l}\right)^2, X_n = \sin\left(\frac{\pi n}{l}x\right), m = 1, 2, 3, \dots$$

$$2. \quad \int_0^l X_m(x)X_n(x)dx = 0 \quad \text{for } m \neq n$$

$$\sin x \sin y = \cos(x - y) - \cos(x + y)$$

$$\sin(my) \sin(ny) = \cos[(m - n)y] - \cos[(m + n)y]$$

$$\int_0^l \sin \frac{m\pi x}{l} \sin \frac{n\pi x}{l} dx \stackrel{y=\pi x/l}{=} \frac{l}{\pi} \int_0^\pi \sin(my) \sin(ny) dy = 0$$

同理 $-T''(t) = c^2 \lambda T(t)$

$$T_n(t) = A_n \cos\left(\frac{c\pi n}{l}t\right) + B_n \sin\left(\frac{c\pi n}{l}t\right)$$

$$3. \quad \text{Fourier 展開解的結構 } u(x, t) = \sum_{n=1}^{\infty} [A_n \cos\left(\frac{nc\pi t}{l}\right) + B_n \sin\left(\frac{nc\pi t}{l}\right)] \sin\left(\frac{n\pi x}{l}\right)$$

$$u(x, 0) = \varphi(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right)$$

$$A_n = \frac{2}{l} \int_0^l \varphi(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

$$u_t(x, 0) = \psi(x) = \sum_{n=1}^{\infty} \frac{n\pi x}{l} B_n \sin\left(\frac{n\pi x}{l}\right)$$

$$B_n = \frac{2}{n\pi x} \int_0^l \psi(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

The natural frequencies are $\omega_n = c\sqrt{\lambda_n} = \frac{c\pi n}{l}, n \in \mathbb{N}$

$\omega_1 = \frac{c\pi}{l}$; principal frequency (fundamental tone 基頻)

overtones(泛音) ; $\omega_n = n\omega_1, n \geq 2$

4. For Neumann boundary condition $u_x(t, 0) = u_x(t, l) = 0$ 比較(1)Dirichlet
(2)Neumann condition 的 eigenfrequencies

特性	Dirichlet 條件	Neumann 條件
端點約束	固定 ($u = 0$)	自由 ($\partial u / \partial x = 0$)
特徵值 λ_n	$\left(\frac{n\pi}{l}\right)^2$ ($n \geq 1$)	$\left(\frac{n\pi}{l}\right)^2$ ($n \geq 0$)
空間特徵函數	$\sin\left(\frac{n\pi x}{l}\right)$	$\cos\left(\frac{n\pi x}{l}\right)$
基頻 ($n = 1$)	$\omega_1 = \frac{c\pi}{l}$	$\omega_1 = \frac{c\pi}{l}$
泛音 ($n \geq 2$)	$n\omega_1$	$n\omega_1$
是否存在零頻 ($n = 0$)	否	是 (靜態解)

(1) Dirichlet 邊界條件(固定端點) :

空間方程解

解得特徵值與特徵函數 :

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, \quad X_n(x) = \sin\left(\frac{n\pi x}{l}\right), \quad n = 1, 2, 3, \dots$$

時間方程解

$$\omega_n = c\sqrt{\lambda_n} = \frac{c\pi n}{l}, n \in N$$

(2) Neumann 邊界條件(自由端點) :

空間方程解

解得特徵值與特徵函數 :

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, \quad X_n(x) = \cos\left(\frac{n\pi x}{l}\right), \quad n = 0, 1, 2, \dots$$

其中 $n = 0$ 對應常數解 $\lambda_0 = 0$ 。

時間方程解

固有頻率 $\omega_n = c\sqrt{\lambda_n} = \frac{cn\pi}{l}, n = 1, 2, 3, \dots$

基頻 $\omega_1 = \frac{c\pi}{l}$ 泛音 $\omega_n = n\omega_1, n \geq 2$

結論：頻譜相同

本質區別

Dirichlet 條件：無零頻解，所有模態均為振動。

Neumann 條件：存在零頻解（ $n=0$ ），允許系統整體平移或靜態變形。

§ S^1 上的一維振動

特徵值問題： $-\frac{d^2u}{dx^2} = \lambda u, u(x+2\pi) = u(x), \lambda \geq 0$

譜的完整描述：

Laplace 算子 Δ 的譜是離散的，譜 $(\Delta) = \{0, 1^2, 2^2, \dots, n^2, \dots\}$

特徵函數 $\{\cos(nx), \sin(nx)\}$ （與 $e^{in\theta}$ 等價，構成 $L^2(S^1)$ 的一組完備正交基。

特徵值 $\lambda_n = n^2$

$\lambda_0 = 0$ ：重數 1 (常數函數)

$\lambda_n = n^2 (n \geq 1)$ ：重數 2 ($\cos(nx)$ 與 $\sin(nx)$ 線性獨立)