§ The Laplacian on a compact Riemannian manifold

§ 3.1 Basic Riemannian Geometry

A connection on M is a map $D: TM \times \Gamma(M) \to TM$

$D_X Y := D(X, Y)$

Lie derivative $L_X(\alpha) := \lim_{t \to 0} \frac{\varphi_t^* \alpha - \alpha}{t}$

Cartan formula $L_X(\alpha) = \iota_X d\alpha + d(\iota_X \alpha)$ for all $\alpha \in \Omega^k(M)$

In particular，for a function f，we get $L_X f = \iota_X df = df(X)$

The Levi-Civita connection is the unique connection on TM such that

(1) D is torsion free

(2) D is compatible with the metric g

For a Riemannian manifold (M, g) of dimension n and for a local chart $\phi: U \subset M \to \mathbb{R}^n$ which coordinates are denoted by $\left(x^1, x^2, \ldots, x^n\right)$, we denote by $\left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \cdots, \frac{\partial}{\partial x^n}\right)$ the associated vector fields. Then we have the following local expression of the Levi-Civita connection:

$$D_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_{k=1}^{n} \Gamma_{ij}^k \frac{\partial}{\partial x_k},$$

where Γ_{ij}^k denote the *Christoffel symbols*.

The geodesic $\nabla_T T = 0$

The exponential map

Theorem 3.1.19 (Hopf–Rinow Theorem). *Let (M, g) be a Riemannian connected manifold. The following conditions are equivalent:*

(i) *The manifold (M, g) is complete.*

(ii) *There exists $x \in M$ such that the map $\exp_x$ is defined on the whole tangent space $T_x M$.*

(iii) *The map $\exp_x$ is defined on the whole tangent space $T_x M$ for all $x \in M$.*

(iv) *Compact sets of M are exactly closed and bounded sets of M.*

Moreover, any of these conditions implies that every pair of points $(x, y) \in M^2$ can be joined by a geodesic curve.

From now on, all Riemannian manifolds are supposed to be complete.

The curvature tensor

$$R(X,Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]} Z$$

For a fixed point x on the manifold M, the *sectional curvature* of a 2-plane $P \subset T_x M$ spanned by a basis X_1, X_2 is the number

$$K_x(P) := \frac{g\,(R\,(X_1, X_2)\,X_2, X_1)}{g\,(X_1 \wedge X_2, X_1 \wedge X_2)} = \frac{R(X_1, X_2, X_1, X_2)}{\|X_1 \wedge X_2\|^2},$$

The Ricci curvature tensor

$Ric_x(X, X) := \sum_{i=1}^{n} R(X, e_i, X, e_i)$ for any point $x \in M$ and any vector $X \in T_x M$,

where $\{e_i\}$ is an orthonormal basis of the vector space $T_x M$ 。

	Ricci curvature	Scalar curvature
R^n	0	0
S^n	(n-1)g	n(n-1)
H^n	-(n-1)g	-n(n-1)

Now, let us introduce the divergence operator. Let X be a $\mathcal{C}^1$ vector field on M. The *divergence* of X is given by

$$\operatorname{div}(X) := \operatorname{tr}(Y \mapsto D_Y X).$$

For a local chart $\phi: U \subset M \to \mathbb{R}^n$ of M and for a vector field

$$X = \sum_{j=1}^{n} X^j \frac{\partial}{\partial x^j},$$

we have

$$\operatorname{div}(X) = \frac{1}{\sqrt{g}} \sum_{i=1}^{n} \frac{\partial\left(\sqrt{g} X^j\right)}{\partial x^j}.$$

§ 3.2 Analysis on manifolds

3.2.1 Distributions on a Riemannian manifold

D(M)：the set of smooth functions with compact suport on M，called the set of test functions。

A distribution on M is a linear form $T: D(M) \to R$, $\varphi \to T\varphi$

Example 3.2.3. Let a be a point M. The *Dirac distribution* δ_a is defined by: for all $\varphi \in \mathcal{D}(M)$,

$$\langle \delta_a, \varphi \rangle_{\mathcal{D}'(M) \times \mathcal{D}(M)} := \varphi(a).$$

Example 3.2.4. A locally integrable function f on M defines a distribution, called the *regular distribution* associated to f, by: for all $\varphi \in \mathcal{D}(M)$

$$\langle T_f, \varphi \rangle_{\mathcal{D}'(M)\times\mathcal{D}(M)} := \int_M f\varphi \, d\mathcal{V}_g .$$

For a distribution $T \in \mathcal{D}'(M)$ and for a local coordinate system $(x^1, x^2, \cdots, x^n)$ we define the partial derivative $\frac{\partial T}{\partial x^i}$ of T by: for all $\varphi \in \mathcal{D}(M)$

$$\left\langle \frac{\partial T}{\partial x^i}, \varphi \right\rangle_{\mathcal{D}'(M)\times\mathcal{D}(M)} := -\left\langle T, \frac{\partial \varphi}{\partial x^i} \right\rangle_{\mathcal{D}'(M)\times\mathcal{D}(M)} .$$

3.2.2 Sobolev spaces on a Riemannian manifold

Lebesgue space $L^2(\mathrm{M}, g) := \{ f : M \to R$ measurable such that $\int_M |f|^2 dV_g < \infty \}$

Soboev space $H^1(M, g) := \overline{C^\infty(M)}$, and $H_0^1(M, g) := \overline{D(M)}$

3.2.3 The Laplacian operator and the Green formula

The Laplace-Beltrami(simply Laplacian)

$\Delta_g(f) :=\mathrm{div}(\nabla \mathrm{f})$

Theorem 3.2.9 (Green's formula aka. Integration by parts formula). *Let (M, g) be an oriented Riemannian manifold and Ω a subset of M with a smooth boundary ∂M. Denote by ν the unit normal vector field to the boundary ∂M. For any $\varphi \in \mathcal{C}^2(M)$ and $\psi \in \mathcal{C}^1(M)$, at least one of which has a compact support, we have:*

$$\int_M \psi \Delta_g \varphi \, d\mathcal{V}_g = \int_{\partial M} (g(\nu, \nabla\varphi)\psi - g(\nu, \nabla\psi)\varphi) \, d\mathcal{A}_g + \int_M \varphi \Delta_g \psi \, d\mathcal{V}_g .$$

Exercise

Verify that $Vol(S^{2n}, can) = \dfrac{(4\pi)^n (n-1)!}{(2n-1)!}$ and $Vol(S^{2n+1}, can) = \dfrac{2(\pi^{n+1})}{n!}$