§ Eigenvalues

§ 01 The wave equation on a string

§ 02 The heat equation

§ 03 The Dirichlet eigenvalue problem

§ 04 The Neumann eigenvalue problem

§ 05 Eigenvalues of Laplace-Beltrami operator on S^2

Using spherical coordinates (θ,ϕ)，the Laplace-Beltrami operator on S^2 is：

$$\Delta_{S^2} = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta}) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} ...(*)$$

We seek eigenfunctions $Y(\theta,\phi)$，and eigenvalues λ such that $\Delta_{S^2}Y = -\lambda Y$

Separation of variables

設$Y(\theta,\phi)=\Theta(\theta)\Phi(\phi)$代入(*)得

$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial\Theta}{\partial\theta})\Phi + \frac{1}{\sin^2\theta}\Theta\frac{\partial^2\Phi}{\partial\phi^2} = -\lambda\Theta\Phi$，同除以$\Theta\Phi$，再同乘以$\sin^2\theta$

$$\frac{\sin\theta}{\Theta}\frac{d}{d\theta}(\sin\theta\frac{d\Theta}{d\theta}) + \lambda\sin^2\theta + \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = 0$$

設$\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = -m^2$，則

$$\begin{cases} \frac{d^2\Phi}{d\phi^2} + m^2\Phi = 0 ...(1) \\ \frac{\sin\theta}{\Theta}\frac{d}{d\theta}(\sin\theta\frac{d\Theta}{d\theta}) + (\lambda\sin^2\theta - m^2)\Theta = 0 ...(2) \end{cases}$$

(1) 式是方位角方程式，(2)式是極角方程式。

由(1) $\Phi(\phi)=e^{im\phi}$，其中 m 是整數(因為 BC $\Phi(\phi+2\pi)=\Phi$)

(2) 的部分，令 $x=\cos\theta$，$\sin\theta=\sqrt{1-x^2}$，$\frac{d}{d\theta} = -\sin\theta\frac{d}{dx} = -\sqrt{1-x^2}\frac{d}{dx} \cdots (3)$

設$P(x)=\Theta(\theta)$ 代入(3) $\frac{d\Theta}{d\theta} = -\sqrt{1-x^2}\frac{dP}{dx}$，$\sin\theta\frac{d\Theta}{d\theta} = -(1-x^2)\frac{dP}{dx}$

計算$\frac{d}{d\theta}(\sin\theta\frac{d\Theta}{d\theta}) = \frac{d}{d\theta}(-(1-x^2)\frac{dP}{dx}) = \sqrt{1-x^2}\frac{d}{dx}((1-x^2)\frac{dP}{dx})$

$$\sin\theta\frac{d}{d\theta}(\sin\theta\frac{d\Theta}{d\theta}) = (1-x^2)\frac{d}{dx}((1-x^2)\frac{dP}{dx})$$

(2)式變成$(1-x^2)\frac{d}{dx}((1-x^2)\frac{dP}{dx}) + (\lambda(1-x^2)-m^2)P = 0$同處以$(1-x^2)$後展開得

$$(1-x^2)\frac{d^2P}{dx^2} - 2x\frac{dP}{dx} + (\lambda - \frac{m^2}{1-x^2})P = 0$$

得到 associated Legendre equation：

$$\frac{d}{dx}((1-x^2)\frac{dP}{dx})+(\lambda-\frac{m^2}{1-x^2})P=0$$

Eigenvalue $\lambda=l(l+1)$ 由 associated Legendre equation 解的存在性決定。

最後得到 eigenfunctions $Y_l^m(\theta,\phi)\propto P_l^m(\cos\theta)e^{im\phi},\lambda_l=l(l+1)$

後記

1. 完整的解為 $Y_l^m(\theta,\phi)=\sqrt{\frac{2l+(l-m)!}{4\pi(1+m)!}}P_l^m(\cos\theta)e^{im\phi}$
2. 經典物理中，重力與靜電場滿足 $\Delta\Phi=0$，當問題約束在球面上時就變成 $\Delta_{S^2}\Phi=0$；例如球面上的熱擴散方程 $\frac{\partial u}{\partial t}=\Delta_{S^2}u$
3. 陀螺運動方程的(1)對稱性 (2)角動量量子化 (3)球面幾何與 Laplace-Beltrami operator 有深層的關係。
4. 陀螺的轉動 vs SO(3)，Laplace-Beltrami operator 是 SO(3)的 Casimir 算子！？