§ Prove the Weyl asymoptotic formula for the Laplacian on a rectangular domain

Let $\Omega = (0,a)\times(0,b)$ be a rectangle in R^2 with Dirichlet boundary conditions。

The eigenvalues of the Laplacian $-\Delta$ are

$\lambda_{m,n} = (\frac{\pi m}{a})^2 + (\frac{\pi n}{b})^2$ for positive integers $m,n \ge 1$ 。

Let $N(\lambda)$ denote the counting function (the number of eigenvalues (counted with multiplicity)less than or equal to λ 。

$$N(\lambda) = \#\{(m,n)\in N^2 : (\frac{\pi m}{a})^2 + (\frac{\pi n}{b})^2 \le \lambda\}$$

The Weyl formula

$\mathrm{N}(\lambda) \sim \frac{|\Omega|}{4\pi}\lambda$ as $\lambda \to \infty$ where $|\Omega| = ab$ is the area of the rectangle。

Prove

1. Counting lattice points

   We want to count the number of integer points in the first quadrant in side the ellipse： $(\frac{\pi m}{a})^2 + (\frac{\pi n}{b})^2 \le \lambda$ (or $\frac{m^2}{a^2} + \frac{n^2}{b^2} \le \frac{\lambda}{\pi^2}$)

2. Approximate by area

   橢圓面積 $= \frac{a\sqrt{\lambda}}{\pi} \times \frac{b\sqrt{\lambda}}{\pi} \times \pi = ab\lambda = |\Omega|\lambda$

   For large λ ，the number of lattice points approaches the area of the ellipse in the first quadrant。

   所以 $\mathrm{N}(\lambda) \sim \frac{|\Omega|}{4\pi}\lambda$ as $\lambda \to \infty$