Prove that the Hermite functions $e_n(x)=(2^n n!\sqrt{\pi})^{-\frac{1}{2}}e^{-\frac{x^2}{2}}H_n(x)$

with $H_n(x)=(-1)^n e^{x^2}\dfrac{d^n}{dx^n}(e^{-x^2})$ form a Hilbert basis of $L^2(R)$

§ 1 正交性

Hermite 多項是滿足 $\int_{-\infty}^{\infty}H_m(x)H_n(x)e^{-x^2}dx=\sqrt{\pi}2^n n!\delta_{mn}$

今考慮 $e_n=\dfrac{1}{\sqrt{2^n n!\sqrt{\pi}}}e^{-x^2/2}H_n(x)$ 因此兩 Hermite 函數的內積為

$\langle e_m,e_n\rangle=\int_{-\infty}^{\infty}e_m(x)e_n(x)dx=...=\delta_{mn}$

§ 完備性(張成空間密度)

To show that the span of $\{e_n\}$ is dense in $L^2(R)$，it suffices to show that if $f\in L^2(R)$

satisfies $\langle f,e_n\rangle=0$ for all n，then f = 0 almost everyhere。

The Hermite functions are of the form $e_n(x)=c_nH_n(x)e^{-x^2/2}$, where $c_n=(2^n n!\sqrt{\pi})^{-1/2}\neq 0$. Thus, the span of $\{e_n\}$ is the same as the span of $\{H_n(x)e^{-x^2/2}\}$. Consider the set $S=\mathrm{span}\{p(x)e^{-x^2/2}\mid p \text{ is a polynomial}\}$, which is identical to the span of the Hermite functions.

Suppose $f\in L^2(\mathbb{R})$ and $\langle f,s\rangle=0$ for all $s\in S$, i.e., for all polynomials p,

$$\int_{-\infty}^{\infty}f(x)p(x)e^{-x^2/2}dx=0.$$

Define $k(x)=f(x)e^{-x^2/2}$. Since $f\in L^2(\mathbb{R})$ and $e^{-x^2/2}$ is bounded (by 1), $k\in L^2(\mathbb{R})$ because

$$\int_{-\infty}^{\infty}|k(x)|^2dx=\int_{-\infty}^{\infty}|f(x)|^2e^{-x^2}dx\leq\sup_x e^{-x^2}\int_{-\infty}^{\infty}|f(x)|^2dx<\infty.$$

The condition becomes

$$\int_{-\infty}^{\infty}k(x)p(x)dx=0\quad\text{for all polynomials } p.$$

It must be shown that $k=0$ almost everywhere. For any continuous compactly supported function ϕ, by the Weierstrass approximation theorem, for any $\epsilon>0$ and any interval $[-A,A]$ containing the support of ϕ, there exists a polynomial q such that

$$\sup_{x\in[-A,A]}|\phi(x)-q(x)|<\epsilon.$$

Then,

$$\left|\int_{-\infty}^{\infty} k(x)\phi(x)dx\right| = \left|\int_{-A}^{A} k(x)\phi(x)dx\right| \le \left|\int_{-A}^{A} k(x)(\phi(x)-q(x))dx\right| + \left|\int_{-A}^{A} k(x)q(x)dx\right|.$$

The first term satisfies

$$\left|\int_{-A}^{A} k(x)(\phi(x)-q(x))dx\right| \le \int_{-A}^{A} |k(x)||\phi(x)-q(x)|dx \le \epsilon \int_{-A}^{A} |k(x)|dx \le \epsilon\sqrt{2A}\|k\|_{L^2(\mathbb{R})}.$$

The second term satisfies, since $\int_{-\infty}^{\infty} k(x)q(x)dx = 0$,

$$\left|\int_{-A}^{A} k(x)q(x)dx\right| = \left|-\int_{|x|>A} k(x)q(x)dx\right| \le \|k\|_{L^2(|x|>A)}\|q\|_{L^2(|x|>A)}.$$

As $A \to \infty$, $\|k\|_{L^2(|x|>A)} \to 0$ because $k \in L^2(\mathbb{R})$. For fixed ϵ and A, $\|q\|_{L^2(|x|>A)}$ is bounded on compact sets, but since ϵ is arbitrary and A can be chosen large, the expression can be made arbitrarily small. Thus, for each fixed ϕ,

$$\left|\int_{-\infty}^{\infty} k(x)\phi(x)dx\right| \le \epsilon\sqrt{2A}\|k\|_{L^2(\mathbb{R})} + \|k\|_{L^2(|x|>A)}\|q\|_{L^2(|x|>A)},$$

and taking $\epsilon \to 0$ and $A \to \infty$ shows that

$$\int_{-\infty}^{\infty} k(x)\phi(x)dx = 0$$

for all continuous compactly supported ϕ. Since such functions are dense in $L^2(\mathbb{R})$, it follows that $k = 0$ almost everywhere.

Therefore, $k(x) = f(x)e^{-x^2/2} = 0$ almost everywhere, so $f = 0$ almost everywhere (since $e^{-x^2/2} \neq 0$). This implies that the only function orthogonal to all elements of S is zero, so S is dense in $L^2(\mathbb{R})$. Hence, the span of the Hermite functions is dense.