§ Let (M_1, g_1) and (M_2, g_2) be two closed Riemannian manifolds。
Find the spectrum and eigenvectors of the product $(M_1 \times M_2, g_1 \times g_2)$

$$\mathrm{Spec}(\Delta_{g_1 \times g_2}) = \{\lambda + \mu \mid \lambda \in \mathrm{Spec}(\Delta_{g_1}),\ \mu \in \mathrm{Spec}(\Delta_{g_2})\}.$$

For each eigenvalue $\nu = \lambda + \mu$ in the spectrum。

The eigenvectors (eigenfunctions) of $\Delta_{g_1 \times g_2}$ are all functions of the form $\phi \otimes \psi$, where:

- ϕ is an eigenfunction of Δ_{g_1} on M_1 with eigenvalue λ,
- ψ is an eigenfunction of Δ_{g_2} on M_2 with eigenvalue μ,
- and $\nu = \lambda + \mu$.

Each such product function $\phi \otimes \psi$ satisfies:

$$\Delta_{g_1 \times g_2}(\phi \otimes \psi) = (\lambda + \mu)(\phi \otimes \psi).$$

The multiplicity of an eigenvalue ν in the product spectrum is the sum of the products of the multiplicities from each manifold:

$$\mathrm{mult}(\nu) = \sum_{\substack{\lambda, \mu \\ \lambda + \mu = \nu}} \mathrm{mult}(\lambda) \cdot \mathrm{mult}(\mu),$$

where $\mathrm{mult}(\lambda)$ is the multiplicity of λ in $\mathrm{Spec}(\Delta_{g_1})$, and $\mathrm{mult}(\mu)$ is the multiplicity of μ in $\mathrm{Spec}(\Delta_{g_2})$.