關於自旋(Spinor)的傳說。，

高中學過純量、向量，大學學到張量，旋量只是一個傳說。

Spinors were originally introduced by Elie Cartan in 1913，and subsequently greatly expanded upon by Hermann Weyl，Richard Brauer and Oswald Veblen。

在描述費米子(Fermions，例如電子、質子、中子)的行為時是不可或缺的。

基本粒子都具有本徵角動量：自旋。

旋量是相對論量子力學、量子場論的基本對象。

但是旋量沒有自洽的經典力學模型對應，旋量是與空間軌道運動無關的轉動生成元。

薛丁格方程式 $\hat{H}\psi = \frac{ih}{2\pi}\frac{\partial \psi}{\partial t}$ 。 Erwin Schrodinger 1887-1961

其中 $\hat{H}$ 是哈密頓算符，ψ 是波函數。

狄拉克方程式 1928 年 $(i\hbar\gamma^{\mu}\partial_{\mu} - mc)\psi(x) = 0$ Paul Dirac 1902-1984

$\psi : R^{1,3} \to C^4$ 稱 Dirac spinor field。 $R^{1,3}$ ：Minkowski space。

γ^{μ} ： 4×4 Dirac gamma matrices。

Otto Stern and Walther Gerlach

1921-1922 年，Otto Stern 與 Walther Gerlach 的實驗證實了電子的自旋(與質量、電荷一樣、是基本粒子的內稟性質。) (1) p.159

George Uhlenbeck and Samuel Goudsmit

1925 年，George Uhlenbeck 與 Samuel Goudsmit 確定電子存在兩自旋態，$\pm\frac{1}{2}$

Each with the units of angular momentum $\hbar=\dfrac{h}{2\pi}$ 。

經典李群中僅正交群有旋量表示。

$SL(2,C)=\{\mathrm{U}=\begin{pmatrix} a & b \\ c & d \end{pmatrix} | a,b,c,d \in C, \det U=1\}$ ，and its Lie algebra $sl(2,C)$ 。

$a,b\in C$ ，then $U=\begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$ $aa^*+bb^*=1$ is unitary(么正)，$U^*=\begin{pmatrix} a^* & -b \\ b^* & a \end{pmatrix}$

$U^*=U^{-1}$ 共軛轉置 conjugate transpose。

$\xi'=U\xi$ spinor transformation。$SL(2,C)$ 稱為自旋變換群。

因此，何謂旋量(spinor)？：

two-component，vector-like quantity with special transformation property，in which rotations and Lorentz boosts are built into the overall(全面的)formalism。

用 Lorentz group theory(Lorentz transformation of rotation and boosts)可以做最簡潔的闡釋。

Dirac 方程

$$i\hbar\frac{\partial}{\partial t}\psi=H\psi=(-i\hbar c\alpha\bullet\nabla+\beta mc^2)\psi$$

其中ψ為含有四個分量的旋量，稱為 Dirac 旋量。

α,β均為 4x4 的矩陣：

$$\alpha_i=\begin{bmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{bmatrix},\ \beta=\begin{bmatrix} I_2 & 0 \\ 0 & -I_2 \end{bmatrix}$$

§ Pauli matrices σ_μ

$$\sigma_0=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\sigma_x=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\sigma_y=\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\sigma_z=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Lorentz algebra

$$[\frac{1}{2}\sigma_x,\frac{1}{2}\sigma_y]=\frac{1}{2}\sigma_x\bullet\frac{1}{2}\sigma_y-\frac{1}{2}\sigma_y\bullet\frac{1}{2}\sigma_x=\frac{1}{2}i\sigma_z$$

Cyclic permutation of x，y，z， $[\frac{1}{2}\sigma_i,\frac{1}{2}\sigma_j]=\frac{1}{2}i\in_{ijk}\sigma_k$

$\in_{ijk}$ is the fully-antisymmetric structure constant of the Pauli algebra with $\in_{123}=1$

Dirac 利用矩陣完成了四維時空中 $\mathrm{E}^2=m^2c^4+c^2p^2$ 的開方，而引入了旋量ψ這一重要概念。1928 年

Q：How to find the solution of the Dirac equation for the hydrogen atom？

§ Lorentz transformations

In three dimension，the counterclockwise rotation of some 3-vector V_k about the z axis by the angle θ is given by $V' = R_z V$，or $\begin{bmatrix} V_x' \\ V_y' \\ V_z' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix}$。

This is converted into four-dimensional form

$$\begin{bmatrix} V_0' \\ V_x' \\ V_y' \\ V_z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} V_0 \\ V_x \\ V_y \\ V_z \end{bmatrix}$$

$$R_x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{bmatrix}, \quad R_y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & 0 & -\sin\theta \\ 0 & 0 & 1 & 0 \\ 0 & \sin\theta & 0 & \cos\theta \end{bmatrix}, \quad R_z = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

And convert the 4×4 rotation matrices into **rotation generators**。

Let's assume an infinitesimal rotation for the matrix R_z，which is

$$R_z(d\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & d\theta & 0 \\ 0 & -d\theta & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I + i \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} d\theta$$

Define $J_z = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ to be the generator of the z-rotation。

$R_z(\theta) = \lim_{n\to\infty}(I + \frac{iJ_z\theta}{n})^n = e^{iJ_z\theta}$ to generate a finite angle θ，$e^{iJ_z\theta} = \cos\theta I + i\sin\theta I$

$$J_x = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{bmatrix}, \quad J_y = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{bmatrix}, \quad J_z = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Lorentz boost：

Lorentz boost is a Lorentz transformation which doesn't involve rotation。

For example，Lorentz boost in the x direction looks like：

Unlike rotations, which can be conveniently described with 3×3 matrices, boosts require 4×4 matrices right from the start (which is why I decided to express rotations with four-dimensional matrices). Imagine a frame of reference passing parallel to a fixed frame in the x-direction with velocity v. Then the two frames are related according to the transformation

$$\begin{bmatrix} t' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix}$$

where

$$\beta = v/c, \quad \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

and where v is the velocity of the primed frame with respect to the unprimed (fixed) frame. The identity $\gamma^2 - \beta^2\gamma^2 = 1$ prompts the convenient identification

$$\cosh\phi = \gamma, \quad \sinh\phi = \beta\gamma$$

where ϕ is the “angle” associated with the boost. We can now proceed exactly as we did before with infinitesimal rotations by considering infinitesimal boosts. We summarize the associated boost generators K_i with

$$K_x = \begin{bmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad K_y = \begin{bmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad K_z = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix} \tag{3.2.1}$$

Like the rotations, the the 4×4 boost transformation matrices can be written simply as $e^{iK\cdot\phi}$. Note that the K_i are now all symmetric, as opposed to the antisymmetry of the (hermitian) rotation generators.

A 4-vector $x^\mu = \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$，and Pauli matrices σ_μ，

$$x^\mu \sigma_\mu = t\sigma_0 + x\sigma_x + y\sigma_y + z\sigma_z = \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix} := H$$

H is called the Hermitian matrix

$H' = UHU^*$，where $U = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$

$$\begin{pmatrix} t'+z' & x'-iy' \\ x'+iy' & t'-z' \end{pmatrix} = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix} \begin{pmatrix} a^* & -b \\ b^* & a \end{pmatrix}$$

Then $t' = (aa^* + bb^*)t = t$，the transformation preserves the time component。

$$\det H = t^2 - x^2 - y^2 - z^2$$

$$x' = \frac{1}{2}(aa + a^*a^* - bb - b^*b^*)\,x - \frac{1}{2}i(aa - a^*a^* + bb - b^*b^*)\,y - (ab - a^*b^*)\,z$$
$$y' = \frac{1}{2}i(aa - a^*a^* - bb - b^*b^*)\,x + \frac{1}{2}(aa + a^*a^* + bb + b^*b^*)\,y - i(ab - a^*b^*)\,z$$
$$z' = (a^*b + ab^*)\,x + i(a^*b - ab^*)\,y + (aa^* - bb^*)\,z$$

If we set $a=\cos\frac{1}{2}\theta, b=i\sin\frac{1}{2}\theta$ ，then

$x'=x, y'=y\cos\theta+z\sin\theta, z'=-y\sin\theta+z\cos\theta$ is the set of Lorentz rotation about x-axis。

Similarly，if we set $a=\cos\frac{1}{2}\theta, b=\sin\frac{1}{2}\theta$ ，then

$x'=x\cos\theta-z\sin\theta, y'=y, z'=x\sin\theta+z\cos\theta$ ，rotation about y-axis。

Setting $a=e^{\frac{1}{2}i\theta}, b=0$ ，then

$x'=x\cos\theta+y\sin\theta, y'=-x\sin\theta+y\cos\theta, z'=z$ ，rotation about the z-axis。

The appearance of half-angles in all this is highly significant. In ordinary vector space, a 360° rotation brings the vector back to itself, but in spinor space a full 720° rotation is needed. In that sense, a spinor is rather like an arbitrary vector lying on a Möbius strip; it has to go around the strip twice to get back where it started.

The unitary transformation matrix H thus depends on the choice of a，b。

§ Representations

The Lorentz algebra of the rotation generators J_i is $[J_x, J_y]=iJ_z$

The Lorentz algebra of the Pauli matrices is $[\frac{1}{2}\sigma_x, \frac{1}{2}\sigma_y]=\frac{1}{2}i\sigma_z$

And its cyclic counterparts。

這裡有兩個李群(Lie algebra)，SO(4)與 SU(2)，有相同的 algebra，表示兩者之間有某種重要的(fundamental)關聯性。

徐一鴻先生(Anthony Zee)說的：二十世紀物理最重要的計算，指的是什麼？

Quantum field theory in a nutshell

etc. Thus, the two matrices J_i and $\frac{1}{2}\sigma_i$ have exactly the same algebra. This cannot be a coincidence; it means there is some kind of fundamental correspondence between the matrices, in spite of the fact that one is orthogonal and 4 × 4 with unit determinant, a group that we call SO(4), and the other is unitary and 2 × 2, also with unit determinant, which is called SU(2). This correspondence is given the representation SO(4) = SU(2), where the equal sign is not to be taken literally. This representation is also called the SO(4)·SU(2) "double cover," perhaps only in the sense that the "double" refers to the fact that the rotation dimension is double that of the spinor dimension. (Note that if I had left the rotation matrices in 3-dimensional form, as many texts do, none of this would make any sense.)

K_i ：generator of infinitesimal boosts

$$K_x = \begin{bmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad K_y = \begin{bmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad K_z = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix}$$

These matrices neither commute nor form a Lorentz algebra，instead we have the commutator $[K_x, K_y] = -iJ_z$，along with its cyclic counterparts。

This identity has two unusual properties。

1. It demonstrates that two boosts in different directions result in a rotation。(a phenomenon reponsible for the Thomas precession of an electron in a magnetic field)
2. There is a minus sign that turns out to be all-important in the overall scheme of things。

What it means is that if we append $\pm i$ to any K_i，we get the commutator

$[\pm K_x, \pm K_y [= i\mathrm{J}_z$，which gives precisely the same algebra as that for the J_i。

Thus，if the J_i are assigned the representation $J_i \to \frac{1}{2}\sigma_i$，we can also assign the similar representation $\pm iK_i \to \pm\frac{1}{2}\sigma_i$，that is $e^{K_i} \to e^{\pm\frac{1}{2}i\sigma_i}$

We can therefore write the complete unitary 2×2 transformation matrix for spinoral rotations and boosts as either of two combined quantities，

$U = e^{\frac{1}{2}i\sigma\bullet\theta - \frac{1}{2}\sigma\bullet\phi} ...(1)$，$U = e^{\frac{1}{2}i\sigma\bullet\theta + \frac{1}{2}\sigma\bullet\phi} ...(2)$

So there are indeed two kinds of spinor：one gets transformed under the unitary matrix in (1)，and the other transforms according (2)，with the overall formlism now denoted as $SO(4) = SU(2) \oplus SU(2)$

The spinor associated with (1) is traditionally called a "right-handed" spinor and given the label φ_R，while the other is a "left-handed" spinor，called φ_L。(these spinors also called "Weyl spinors")

One of the amazing facts is that all neutrinos in the universe are left handed，their spinor descriptions are of the left-handed type。

Weyl spinors $\psi = \begin{bmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{bmatrix} = \begin{bmatrix} \varphi_R \\ \varphi_L \end{bmatrix}$。

Carl David Anderson 1905-1991 發現正電子 1932 年。

Dirac' s ispinor was found to preserve parity under the sign reversal operation $\psi(x,t) \to \psi(-x,t)$

Far more importantly，the spinor φ_R effectively represents the spin-up and spin-down components of an ordinary electron，while φ_L represents the spin-up and spin-down components of an anti-electron。(known as a positron)

Dirac' s work thus predicted the existence of antimatter。

Dirac' s relativistic electron equation also explained electron spin as a form of intrinsic angular momentum called S。

Thus，the angular momentum L of an electron alone is not conserved，instead，it is L+S that is conserved。

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2. 物理學家用微分幾何 侯伯元 侯伯宇 Ch12 旋量 自旋流形
3. A Child's Guide to Spinors William O. Straub
4. Differential Geometry in Physics Gabriel Lugo 8.2 Lorentz Group

後記

1770 年 Euler 的工作：

$V = x\vec{i} + y\vec{j} + z\vec{k}$ ，$V \bullet V = x^2 + y^2 + z^2$

$(x^2+y^2+z^2)I_2 = W^2$ 開方得 $W = x\sigma_1 + y\sigma_2 + z\sigma_3 = \begin{bmatrix} z & x-iy \\ x+iy & -z \end{bmatrix}$,

$\det W = -(x^2+y^2+z^2)$

將整個空間轉動，令轉軸平行於(m,n,l)向量，轉角$\theta = 2\arctan\frac{1}{2}\sqrt{m^2+n^2+l^2}$

利用 W，Euler 發現 R^3 中轉動群的一種新表示：

$$W' = \frac{UWU^*}{1+\frac{1}{4}(m^2+n^2+l^2)}, U = \begin{bmatrix} 1+\frac{i}{2}l & \frac{1}{2}(im+n) \\ \frac{1}{2}(im-n) & 1-\frac{i}{2}l \end{bmatrix}$$

當θ轉動為4π時 W 轉回原位，而這時向量 V 轉兩周，W 為向量 V 的二重覆蓋。

Euler 所找到的這種轉動群的新表示，即轉動群 O(3)的旋量表示。

[物理學家用微分幾何] 侯伯元　p.353

後面是 1938 年 Elie Cartan 利用零模向量引入旋量…~p.369

§ 12.2 時空的 Lorentz 變換與自旋變換 旋量張量代數

M^4：4-dim Mikowski spacetime 旋量分析特別有效和方便

在M^4中選一組 orthonormal basis $\{e_\mu\}_1^4$，$(e_\mu, e_\nu) = \eta_{\mu\nu}$

$$\eta_{\mu\nu} = \begin{bmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}，ds^2 = \eta_{\mu\nu} dx_\mu dx^\nu ，\mu, \nu = 0,1,2,3$$

任意向量$v = v^i e_i = te_0 + xe_1 + ye_2 + ze_3$

$$(v,v) = |v|^2 = t^2 - x^2 - y^2 - z^2$$

通過原點的光椎上的點(零模向量)，其座標(t,x,y,z)滿足$t^2 - x^2 - y^2 - z^2 = 0$

可用光椎與 t=1 超面的交集ζ^+來表示將來零模方向空間，其座標滿足

$$x^2 + y^2 + z^2 = 1$$

空間同胚於二為球面S^2，稱為(逆)天體球(celestial sphere)ζ^+，球ζ^+表示觀察者

視界，又稱為天空映射(skymapping)。ζ^+上各點座標(x,y,z)可用一複數ζ表示

$$\zeta = \frac{x+iy}{t-z} = \frac{t+z}{x-iy} = e^{i\varphi} \cot\frac{\theta}{2} \quad ...(*)$$

相當於球面S^2以北極(z=1)為極點對赤道面(z=0)作極設投影，利用

$\zeta\bar{\zeta} = \dfrac{x^2+y^2}{(t-z)^2} = \dfrac{t+z}{t-z}$ 可解得(*)的逆變換

$$\frac{z}{t}=\frac{\zeta\bar{\zeta}-1}{\zeta\bar{\zeta}+1},\frac{x}{t}=\frac{\zeta+\bar{\zeta}}{\zeta\bar{\zeta}+1},\frac{iy}{t}=\frac{\zeta-\bar{\zeta}}{\zeta\bar{\zeta}+1}$$

為了表示整個將來光椎方向，應將ζ用兩個複數比表示$\zeta=\xi/\eta$

這樣可解得光椎上的零模向量座標

$$z=\frac{1}{\sqrt{2}}(\xi\bar{\xi}-\eta\bar{\eta})\text{，}x=\frac{1}{\sqrt{2}}(\xi\bar{\eta}+\eta\bar{\xi})\text{，}t=\frac{1}{\sqrt{2}}(\xi\bar{\xi}+\eta\bar{\eta}),y=\frac{1}{i\sqrt{2}}(\xi\bar{\eta}-\eta\bar{\xi})$$

可將零模向量的座標排成2×2 Hermitian matrix，其行列式為零。

$$H=\frac{1}{\sqrt{2}}\begin{bmatrix} t+z & x+iy \\ x-iy & t-z \end{bmatrix}=\begin{bmatrix} \xi\bar{\xi} & \xi\bar{\eta} \\ \eta\bar{\xi} & \eta\bar{\eta} \end{bmatrix}=\begin{bmatrix} \xi \\ \eta \end{bmatrix}\begin{bmatrix} \bar{\xi} & \bar{\eta} \end{bmatrix}$$

即$\varphi=\begin{bmatrix} \xi \\ \eta \end{bmatrix},H=\varphi\varphi'$ 零模向量的"開方"為旋量φ

(ξ,η)的任意複線性變換可導致(t,x,y,z)的實現性變換

$$\begin{bmatrix} \xi' \\ \eta' \end{bmatrix}=\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}\begin{bmatrix} \xi \\ \eta \end{bmatrix},\alpha,\beta,\gamma,\delta\in C,\alpha\delta-\beta\gamma=1$$

即旋量變換$\varphi'=A\varphi$，detA=1

自旋變換群$SL(2,C)=\{A=\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}\big|\det A=1\}$ p.370

§ 12.3 Dirac 旋量 Weyl 旋量 純旋量 各維旋量的矩陣表示結構

§ 12.4 Majorana 表象

§ 12.5 自旋結構與自旋流形 Spin 結構

§ 12.6 自旋結構的聯絡 Dirac 算子 Weitzenbock 公式