LECTURE 11: THE FROBENIUS THEOREM

1. Distributions

Suppose M is an n-dimensional smooth manifold. We have seen that any smooth vector field X on M can be integrated locally near any point to an integral curve. Moreover,

- If $X_p = 0$, then the corresponding integral curve is the constant curve $\gamma_p(t) \equiv p$.
- If $X_p \neq 0$, then the corresponding integral curve $\gamma_p(t)$ is a 1-dimensional curve near p.

In what follows we would like to some higher dimensional analogue of this fact.

Definition 1.1. A k-dimensional *distribution* $\mathcal{V}$ on M is a map which assigns to every point $p \in M$ a k-dimensional vector subspace $\mathcal{V}_p$ of T_pM. $\mathcal{V}$ is called *smooth* if for every $p \in M$, there is a neighborhood U of p and smooth vector fields $X_1, \cdots, X_k$ on U such that for every $q \in U$, $X_1(q), \cdots, X_k(q)$ are a basis of $\mathcal{V}_q$. (In particular, $X_i(q) \neq 0$ for all $1 \leq i \leq k$.)

Remarks. (1) In what follows, all distributions will be smooth.
(2) We say a vector field X belongs to a distribution $\mathcal{V}$ if $X_p \in \mathcal{V}_p$ for all $p \in M$.
(3) By definition, a k-dimensional distribution is rank k sub-bundle of TM.

Definition 1.2. Suppose $\mathcal{V}$ is a k-dimensional distributions on M. An immersed submanifold $N \subset M$ is called an *integral manifold* for $\mathcal{V}$ if for every $p \in N$, the image of $d\iota_N : T_pN \to T_pM$ is $\mathcal{V}_p$. We say the distribution $\mathcal{V}$ is *integrable* if through each point of M there exists an integral manifold of $\mathcal{V}$.

Example. Any non-vanishing vector field X is a 1-dimensional distribution. The image of any integral curve of X is an integral manifold.

Example. The vector fields $\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_k}$ span a k-dimensional distribution $\mathcal{V}$ in $\mathbb{R}^n$. The integral manifolds of $\mathcal{V}$ are planes that are defined by the system of equations

$$x^i = c^i \quad (k+1 \leq i \leq n).$$

Remark. An integral manifold need not to be an embedded submanifold of M. For example, consider $M = S^1 \times S^1 \subset \mathbb{R}^2_x \times \mathbb{R}^2_y$. Fix any irrational number a, the integral manifold of the non-vanishing vector field

$$X^a = (x^2\frac{\partial}{\partial x^1} - x^1\frac{\partial}{\partial x^2}) + a(y^2\frac{\partial}{\partial y^1} - y^1\frac{\partial}{\partial y^2})$$

is a dense "curve" in M. (However, it is an immersed submanifold.)

Although any 1-dimensional distribution is integrable, a higher dimensional distribution need not to be integrable.

Example. Consider the smooth distribution $\mathcal{V}$ on $\mathbb{R}^3$ spanned by two vector fields

$$X_1 = \frac{\partial}{\partial x^1} + x^2 \frac{\partial}{\partial x^3}, \quad X_2 = \frac{\partial}{\partial x^2}.$$

I claim that there is no integral manifold through the origin. In fact, if $\mathcal{V}$ is integrable, then the integrable manifold N of $\mathcal{V}$ containing the origin must also contain the integrable curve of X_1 passing the origin, which is a piece of the x^1-axis, i.e. N contains all points of the form $(t, 0, 0)$ for $|t| < \varepsilon$. Also N must contain the integral curves of the vector field X_2 passing all these points $(t, 0, 0)$. It follows that for each $|t| < \varepsilon$, N contains a small piece of line segment parallel to the x^2-axis, i.e. N contains for each $|t| < \varepsilon$ all points of the form $(t, s, 0), |s| < \delta_t$. In other words, N contains a piece of the x^1-x^2 plane that contains the origin. This is a contradiction, because the vector $\frac{\partial}{\partial x^1}$ is a tangent vectors of this piece of plane but is not in $\mathcal{V}_p$ for any $p \neq 0$.

We are interested in the conditions to make a distribution integrable. A necessary condition is easy to find. In fact, we have

Theorem 1.3. *If a distribution $\mathcal{V}$ is integrable, then for any two vector fields X and Y belonging to $\mathcal{V}$, their Lie bracket $[X, Y]$ belongs to $\mathcal{V}$ also.*

Proof. Fix any $p \in M$, suppose $\iota : N \hookrightarrow M$ is an integrable manifold of $\mathcal{V}$. Since N is an immersed submanifold of M, one can "shrink" N so that $\iota(N)$ is in fact an embedded submanifold of M. Now suppose X, Y are vector fields belonging to $\mathcal{V}$, then the restrictions $X|_N, Y|_N$ to N are vector fields that are tangent to the submanifold N. By definition, X_N is ι related to X and $Y|_N$ is ι-related to Y. It follows that $[X|_N, Y|_N]$ is ι-related to $[X, Y]$. In other words, $[X, Y]_p = d\iota_p([X|_N, Y|_N]_p)$ is tangent to N also. It follows that for any $p \in M$, $[X, Y]_p \in \mathcal{V}_p$. So $[X, Y]$ belongs to $\mathcal{V}$ also. □

Definition 1.4. A distribution $\mathcal{V}$ is *involutive* if it satisfies the following *Frobenius condition*: If $X, Y \in \Gamma^\infty(TM)$ belong to $\mathcal{V}$, so is $[X, Y]$.

Example. Any 1 dimensional distribution is involutive since $[fX, gX]$ is a multiple of X.

Example. The k-dimensional distribution spanned by $\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_k}$ is involutive.

Example. The distribution $\mathcal{V}$ spanned by

$$X_1 = \frac{\partial}{\partial x^1} + x^2 \frac{\partial}{\partial x^3}, \quad X_2 = \frac{\partial}{\partial x^2}$$

is not involutive, since

$$[X_1, X_2] = -\frac{\partial}{\partial x^3}$$

is not in $\mathcal{V}$.

Example. Let $f : M \to N$ be a submersion. Then the distribution $\mathcal{V}$ with $\mathcal{V}_p = \mathrm{Ker}(df_p)$ is involutive. In fact, if X, Y are vector fields belonging to $\mathcal{V}$, then $df_p(X_p) = df_p(Y_p) = 0$, i.e. both X and Y are f-related to the zero vector field on N. It follows that $df([X, Y]_p) = 0$. It is easy to see that $\mathcal{V}$ is also integrable. In fact, the integrable manifold passing $p \in M$ is the submanifold $f^{-1}(f(p))$.

2. The Frobenius Theorem

It turns out that the Frobenius condition is not only necessary but also sufficient for a distribution to be integrable.

Theorem 2.1 (Global Frobenius Theorem)**.** *Let $\mathcal{V}$ be an involutive k-dimensional distribution. Then through every point $p \in M$, there is a unique maximal connected integral manifold of $\mathcal{V}$.*

Example. Consider the distribution $\mathcal{V}$ on $\mathbb{R}^3$ spanned by

$$X_1 = x^1 \frac{\partial}{\partial x^2} - x^2 \frac{\partial}{\partial x^1}, \quad X_2 = \frac{\partial}{\partial x^3}$$

on $M = \mathbb{R}^3 - \{x^1 = x^2 = 0\}$. Since $[X_1, X_2] = 0$, $\mathcal{V}$ is involutive. What is its integral manifold? Well, let's first compute the integral curves of X_1 and X_2. Through any point (x^1, x^2, x^3), the integral curves of X_1 are circles in the x^3-plane with origin the center, and the integral curves of X_2 are the lines that are parallel to the x^3-axis. Note that the integral manifold passing (x^1, x^2, x^3) of the distribution should contains all points of the form $\varphi_t^{X_1}(\varphi_s^{X_2}(x^1, x^2, x^3))$ for all t, s. In our case, this is the cylinders centering at the x^3-axis.

We first prove the following local version: Any involutive distribution is integrable, i.e. locally near each point one can find an integrable manifold.

Theorem 2.2 (Local Frobenius Theorem)**.** *Let $\mathcal{V}$ be an involutive k-dimensional distribution. Then for every $p \in M$, there exists a coordinate patch $(U, x^1, \cdots, x^n)$ centered at p such that for all $q \in U$, $\mathcal{V}_q = \mathrm{span}\{\frac{\partial}{\partial x^1}(q), \cdots, \frac{\partial}{\partial x^k}(q)\}$.*

We need the following lemma whose proof is left as an exercise.

Lemma 2.3. *Let X be a smooth vector field on M. If $p \in M$ such that $X_p \neq 0$, then there exists a local chart $(U, x^1, \cdots, x^n)$ near p such that $X = \frac{\partial}{\partial x^1}$ on U.*

Proof of the Local Frobenius Theorem: By the lemma, this is true for $k = 1$. Suppose the theorem holds for $k-1$ dimensional distributions. Let $\mathcal{V}$ be an k dimensional distribution spanned by $X_1, X_2, \cdots, X_k$. Suppose $\mathcal{V}$ is involutive, i.e.

$$[X_i, X_j] \equiv 0 \mod (X_1, \cdots, X_k), \quad 1 \leq i, j \leq k.$$

Use the previous lemma, there exits a local chart $(U; y^1, \cdots, y^n)$ near p such that $X_k = \frac{\partial}{\partial y^k}$. For $1 \leq i \leq k-1$ let

$$X_i' = X_i - X_i(y^k) X_k,$$

then $X_i'(y^k) = 0$ for $1 \leq i \leq k-1$, and $X_k(y^k) = 1$. Note that the vector fields $X_1', \cdots, X_{k-1}', X_k$ still span $\mathcal{V}$. Moreover, if we denote

$$[X_i', X_j'] = a_{ij} X_k \mod (X_1', \cdots, X_{k-1}'), \quad 1 \leq i, j \leq k-1,$$

then applying both sides to the function y^k, we see $a_{ij} = 0$ for all $1 \leq i, j \leq k-1$. In other words, the $k-1$ dimensional distribution

$$\mathcal{V}' = \mathrm{span}\{X_1', \cdots, X_{k-1}'\}$$

is involutive. So there is a local chart $(U, z^1, \cdots, z^n)$ near p such that $\mathcal{V}'$ is spanned by $\{\frac{\partial}{\partial z^1}, \cdots, \frac{\partial}{\partial z^{k-1}}\}$. Since each $\frac{\partial}{\partial z^i}$ is a linear combination of X'_j for $1 \le i, j \le k-1$, we conclude $\frac{\partial}{\partial z^i}(y^k) = 0$.

Now denote

$$[\frac{\partial}{\partial z^i}, X_k] = b_i X_k \mod (\frac{\partial}{\partial z^1}, \cdots, \frac{\partial}{\partial z^{m-1}}).$$

Apply both sides to the function y^k, we see $b_i = 0$ for all i. So we can write

$$[\frac{\partial}{\partial z^i}, X_k] = \sum_{j=1}^{k-1} C_j^i \frac{\partial}{\partial z^j}.$$

Suppose $X_k = \sum_{j=1}^n \xi_j \frac{\partial}{\partial z_j}$. Insert this into the previous formula, we see

$$\frac{\partial \xi_j}{\partial z_i} = 0, \quad 1 \le i \le k-1, k \le j \le n.$$

In other words, for $j \ge k$, $\xi_j = \xi_j(z^k, \cdots, z^n)$. Let

$$X'_k = \sum_{j=k}^n \xi_j \frac{\partial}{\partial z^j}.$$

Then $\{\frac{\partial}{\partial z^1}, \cdots, \frac{\partial}{\partial z^{k-1}}, X'_k\}$ still span $\mathcal{V}$. Finally according to the pervious lemma again, there is a local coordinate change from $(z^1, \cdots, z^k, \cdots, z^n)$ to $(x^1, \cdots, x^k, \cdots, x^n)$ with $x^i = z^i$ for $1 \le i \le k-1$, such that $X'_k = \frac{\partial}{\partial x^k}$. This completes the proof. □

Sketch of proof of the Global Frobenius theorem: For any $p \in M$, let

$$N_p = \{q \in M \mid \exists \text{ a piecewise smooth integral curve in } \mathcal{V} \text{ jointing } p \text{ to } q\}.$$

We claim that N_p is the maximal connected integral manifold of $\mathcal{V}$ containing p.

The manifold structure is defined as follows: for any $q \in N_p$, there is a coordinate patch $(U, x^1, \cdots, x^n)$ centering at q such that $\mathcal{V} = \text{span}\{\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^k}\}$ in U. For each small ε, let

$$W_\varepsilon = \{w \in U \mid (x^1)^2(w) + \cdots + (x^k)^2(w) \le \varepsilon, x^{k+1}(w) = \cdots = x^n(w) = 0\}.$$

Then any point $w \in W_\varepsilon$ can be joint to p by the integral curve

$$\gamma(t) = t(x_1(w), \cdots, x_k(w), 0, \cdots, 0).$$

So $W_\varepsilon \subset N_p$. Let

$$\varphi : W_\varepsilon \to B^k(\varepsilon) \subset \mathbb{R}^k, \quad w \mapsto (x_1(w), \cdots, x_k(w)).$$

Now we define the topology on N_p by giving it the weakest topology such that all these W_ε's are open. The atlas on N_p is defined to be the set of charts $(\varphi, W, B_k(\varepsilon))$. One can check that N_p is a manifold with this given atlas. For more details, c.f. Warner, pg.48-49. □