DIFFERENTIAL GEOMETRY AND THE UPPER HALF PLANE AND DISK MODELS OF THE LOBACHEVSKI or HYPERBOLIC PLANE

Book 1

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1. INTRODUCTION

This module has a number of goals which are not entirely consistent with one another. Our first objective is to develop the initial stages of Lobachevski geometry in as natural a manner as possible given tools which are commonly available. These are elementary complex analysis and elementary differential geometry. The second goal is to use Lobachevski geometry as an example of a Riemannian Geometry. In some sense it is a maximally simple example since it is not a surface embeddable in 3 space but it is still two dimensional and topologically trivial. Hence it can serve as a sandbox for learning concepts such as geodesic curvature and parallel translation in a context where these concepts are not totally trivial but still not very difficult. Third, by considering the upper half plane model and the disk model we have a good example of how easy things can be when placed in an appropriate setting. We also have a non-trivial and splendidly useful example of an isometry.

The novice at Lobachevski geometry would profit from looking at the intoroductory material which shows pictures of the upper half plane model (Book 0).

A word of caution. Much of the material I develop here can be developed in a more elementary manner, and some persons, for example philosophy students, might be more comfortable going that route, in which case there are many good books available. Some of the material, for example the geodesic curvature of horocycles and equidistants, could also be so developed (I think) but it would require developing special equipment on site, so the differential geometry approach might be more natural for this material.

My primary reason for presenting this material is to have an easily available reference for myself and my students. Since the material is not intended for publication the style is informal and certain niceties have been neglected. However, I believe the material to be accurate, even if occasionally there are missing details. It is important to understand the reference goal of this module; occasional sections compute things like Christoffel symbols so that I have them available when I want them. Other sections, for example the section on Hyperbolic functions, contain well known material which is once again included

[1] 8 Jan 08; Many typos have been fixed and each section now starts on a new page.

for reference purposes. The user should skim or skip these sections which are rather tedious to read.

I have consulted almost no references for this work, relying on my memory of the material from many years ago. Hence there will undoubtedly be places where the development is suboptimal. I intend to fix this given time.

I would not have bothered with this project if it were not fun and I hope you users will also find it fun. However, without some knowledge of elementary differential geometry and complex variables I'm afraid it won't be much fun.

This module has been divided into Books in order to keep the PDFs from becoming too long for convenience.

Capitalization of words is non-standard, and used to lightly emphasize the words or make them stand out in the text. And remember, if you find the material incomprehensible, exasperating, dull or turgid, you didn't pay much for it.

2. A BIT OF HISTORY

Lobachevski geometry developed out of a desire to fix what was perceived as a flaw in Euclid's presentation of Euclidean geometry. Euclid's first four postulates are very simple and the fifth is very complicated. It seemed reasonable to attempt to derive the fifth from the other four. Many people, Greeks, Arabs, Europeans, claimed to have done this but they were always mistaken, because it cannot be done. The usual method was to slip in some equivalent of postulate five without realizing it, for example the existence of two similar but not congruent triangles.

One of the methods used was to assume more than one parallel to a point could be drawn through a given line and reach a contradiction. This was the method used by Sacchieri in 1733. Sacchieri developed a good deal of Lobachevski geometry without finding the sought for contradiction but in the end backed out and disowned his discovery. His social situation was complex and his final rejection of his own discovery may have been a political decision rather than his honest conclusion.

From this point on the question of who influenced whom is difficult to answer.

Lambert (c 1766) was the next person to show an interest in our subject and developed hyperbolic trigonometry (sinh and cosh) as part of his investigations. He looked at the geometry of a sphere with complex radius iR, and this is indeed one entry point to Lobachevski geometry but he did not follow up his discoveries very far or did not publish them. The work of Sacchieri and Lambert seems to have been ignored. Sacchieri's book seems to have always been rare but Lambert's work may have influenced people later, since it was available.

The first person to figure it all out was Gauss, and he kept it quiet, since he knew there would be controversy if he published it. Both Janos Bolyai and Lobachevski have connections to Gauss, Bolyai was the son of a close friend Farkas (Wolfgang) Bolyai and Lobachevski's teacher, Bartels, was also a lifelong friend of Gauss. Thus in both cases leakage of information, perhaps no more than Gauss's belief in an alternative to Euclidean geometry and maybe not even that much, is possible.

Gauss's students Mindung and Taurinus developed a sort of Lobachevski geometry using the pseudosphere. This was simply a constant negative curvature surface but it is not a model of the entire Lobachevski plane. It is, however, eminently noncontroversial and in my opinion of modest importance. Others hold different views on this point.

Bolyai and Lobachevski were the first to assert publicly that they had invented a new geometry for the plane and that it was well worth studying. Bolyai was disappointed by the lack of fame and fortune that followed his discoveries and became a military officer. Lobachevski alone had the courage and the fortitude to develop the theory into a large body of material. He accomplished this in near total isolation at the University of Kazan, where the local population was Tatar and remarkably indifferent to the theory of parallels. In later years he had some correspondence with Gauss who realized the importance of the work but, as usual, could not bring himself to point it out publicly which would have been of enormous help to Lobachevski. (The only person Gauss seems to have helped was his mother.) Lobachevski published several books on his geometry, which were largely ignored because in the opinion of philosophers and others Lobachevski geometry was inconsistent; the contradiction was lurking right around the next corner.

Finally about 1860 Beltrami was able to construct the Upper Half Plane (UHP) model of Lobachevski geometry which proved beyond all doubt that it was both consistent and interesting. Sadly, by this time Lobachevski was dead.

Over time Lobachevski geometry and the UHP model and their generalizations have taken an important place in higher mathematics, partly due to the close connections with modular functions. So of the three sisters, Spherical, Plane and Lobachevski geometry, it is the despised youngest that has turned out to be critical for the future of mathematics. John Stillwell has called this our Cinderella story.

3. REVIEW OF HYPERBOLIC FUNCTIONS

In this section we do a brief review of $\sinh(x)$, $\cosh(x)$, and other hyperbolic functions. Skip to the next section if this material is familiar to you. We will be using this material from time to time throughout this module. Hyperbolic functions were invented by Lambert in 1766 in a context equivalent to Lobachevski geometry

The definitions are

$$\begin{aligned}
\sinh(x) &= \frac{e^x - e^{-x}}{2} \\
\cosh(x) &= \frac{e^x + e^{-x}}{2} \\
\tanh(x) &= \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \\
\coth(x) &= \frac{1}{\tanh(x)} \\
\operatorname{sech}(x) &= \frac{1}{\cosh(x)} \\
\operatorname{csch}(x) &= \frac{1}{\sinh(x)}
\end{aligned}$$

Clearly

$$\begin{aligned}
e^x &= \cosh(x) + \sinh(x) \\
e^{-x} &= \cosh(x) - \sinh(x)
\end{aligned}$$

Note that for real x we have $\sinh(0) = 0$, $\cosh(0) = 1$, and $\sinh(x)$ is an odd function and $\cosh(x)$ is an even function. Also note that $|\sinh(x)| < \cosh(x)$ and thus $|\tanh(x)| < 1$. To get the fundamental formulas, the addition formulas, we use

$$\begin{aligned}
e^x e^{-y} &= (\cosh(x) + \sinh(x))(\cosh(y) - \sinh(y)) \\
&= (\cosh(x)\cosh(y) + \sinh(x)\cosh(y) - \cosh(x)\sinh(y) - \sinh(x)\sinh(y) \\
e^{-x} e^{y} &= (\cosh(x) - \sinh(x))(\cosh(y) + \sinh(y)) \\
&= (\cosh(x)\cosh(y) - \sinh(x)\cosh(y) + \cosh(x)\sinh(y) - \sinh(x)\sinh(y) \\
\frac{1}{2}(e^{x-y} - e^{-x+y}) &= \sinh(x)\cosh(y) - \cosh(x)\sinh(y) \\
\sinh(x-y) &= \sinh(x)\cosh(y) - \cosh(x)\sinh(y) \\
\frac{1}{2}(e^{x-y} + e^{-x+y}) &= \cosh(x)\cosh(y) - \sinh(x)\sinh(y) \\
\cosh(x-y) &= \cosh(x)\cosh(y) - \sinh(x)\sinh(y)
\end{aligned}$$

and hence, using the odd/even properties again,

$$\sinh(x+y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y)$$

$$\begin{aligned}\cosh(x+y) &= \cosh(x)\cosh(y)+\sinh(x)\sinh(y)\\ \tanh(x\pm y) &= \frac{\tanh(x)\pm\tanh(y)}{1\pm\tanh(x)\tanh(y)}\end{aligned}$$

Now if $y=-x$ we get

$$\begin{aligned}\cosh(0) &= \cosh(x-x)=\cosh(x)\cosh(x)-\sinh(x)\sinh(x)\\ 1 &= \cosh^2(x)-\sinh^2(x)\end{aligned}$$

which can also be gotten from the formula for $e^x e^{-y}$ by setting $x=y$. By dividing the last equaton by $\cosh^2(x)$ or $\sinh^2(x)$ we get

$$\begin{aligned}\operatorname{sech}^2(x) &= 1-\tanh^2(x)\\ \operatorname{csch}^2(x) &= \coth^2(x)-1\end{aligned}$$

Next doubling formulas

$$\begin{aligned}\sinh(2x) &= 2\sinh(x)\cosh(x)\\ \cosh(2x) &= \cosh^2(x)+\sinh^2(x)\\ &= 2\cosh^2(x)-1\\ &= 1+2\sinh^2(x)\end{aligned}$$

and halving formulas

$$\begin{aligned}\sinh\left(\frac{1}{2}x\right) &= \sqrt{\frac{\cosh(x)-1}{2}}\\ \cosh\left(\frac{1}{2}x\right) &= \sqrt{\frac{\cosh(x)+1}{2}}\end{aligned}$$

Now we look at the inverse functions which we will write $\operatorname{argsinh}(s)$, $\operatorname{argcosh}(s)$ etc. Let $x=\operatorname{argsinh}(s)$. Then we have

$$\begin{aligned}x &= \operatorname{argsinh}(s)\\ \sinh(x) &= s\\ \frac{1}{2}(e^x-e^{-x}) &= s\\ e^{2x}-2se^x-1 &= 0\\ e^x &= \frac{1}{2}\big(2s\pm\sqrt{4s^2+4}\big)=s\pm\sqrt{s^2+1}\\ \operatorname{argsinh}(s) &= \ln(s\pm\sqrt{s^2+1}) \qquad \text{(Plus sign for real output)}\end{aligned}$$

If x is to be real we must replace $\pm$ by $+$ in the last formula Similarly we have for $\operatorname{argcosh}(x)$

$$\frac{1}{2}(e^x+e^{-x}) = s$$

$$\begin{aligned}
e^{2x} - 2se^x + 1 &= 0 \\
e^x &= s \pm \sqrt{s^2 - 1} \\
\operatorname{argcosh}(s) &= \ln(s \pm \sqrt{s^2 - 1})
\end{aligned}$$

where in this case we do get two real values provided that $s \geq 1$ and *no* real values of $s < 1$. Note also that $s - \sqrt{s^2 - 1} = 1/(s + \sqrt{s^2 - 1})$ so the two values are negatives of one another, as expected for an even function.

For our purposes the most important of the inverse functions is argtanh(s) which will be important in the formula for distance in the Disk Model of Lobachevski Geometry. We derive this formula which we notice lacks the square root characteristic of argsinh and argcosh and instead has a Möbius transformation in it

$$\begin{aligned}
x &= \operatorname{argtanh}(s) \\
\tanh(x) &= s \\
\frac{e^x - e^{-x}}{e^x + e^{-x}} &= s \\
e^{2x} - 1 &= s(e^{2x} + 1) \\
(1 - s)e^{2x} &= 1 + s \\
e^{2x} &= \frac{1 + s}{1 - s} \\
\operatorname{argtanh}(s) &= x = \frac{1}{2} \ln \frac{1 + s}{1 - s}
\end{aligned}$$

This formula will produce real output if and only if $|s| < 1$.

The derivative formulas

$$\begin{aligned}
\frac{d}{dx} \sinh(x) = \cosh(x) \\
\frac{d}{dx} \cosh(x) = \sinh(x) \\
\frac{d}{dx} \tanh(x) = \operatorname{sech}(x)
\end{aligned}$$

are trivial to derive.

4. METRIC COEFFICIENTS, CHRISTOFFEL SYMBOLS AND THE RIEMANN CURVATURE TENSOR

The Upper Half Plane (UHP) model is parametrized by the complex variable z with $\Im(z) > 0$ or by the x and y of $z = x + iy$. Straight lines are defined to be either vertical Euclidean straight lines or semicircles perpendicular to the x-axis.

Because the UHP is not a surface isometrically embedded in a larger three space we cannot proceed as, for example, we do in the Sphere Module. Here we must use the techniques of Riemannian Geometry.

This section is completely computational; I wanted to make the Christoffel symbols and Riemann curvature tensor as well as the connection and curvature forms available for later when I want them. It would certainly be possible to skip this section and return to it when necessary.

For computational purposes remember that $u^1 = x$ and $u^2 = y$.

A Lobachevski plane comes with a natural unit of length, which is related to the Gaussian Curvature. We build it into the metric for the differential geometry of the space as the positive parameter a. The characteristic length is then $1/a$, which has the dimension of a length. It will turn out in the sequel that the Gaussian Kurvature $K = -a^2$. Hence a^2 fits in the position of $K = 1/R^2$ for the sphere, and if we put $a = 1/(iR)$ we see what Lambert already saw in 1760; that the Lobachevski plane is formally similar to a sphere of radius iR instead of R. This allowed J. H. Lambert to develop the hyperbolic functions in analogy to spherical trigonometry and to predict the form Lobachevski trigonometry would take.

For almost all purposes we could set $a = 1$, but we will keep our options open. The parameter a is analogous to $1/R$ for the sphere and setting $a = 1$ is analogous to using a unit sphere.

The metric is given by

$$ds^2 = \frac{dx^2 + dy^2}{a^2\, y^2}$$

It is not fair to give both a definition of straight lines *and* a metric, but we will verify later that the straight lines are indeed geodesics, so there is no problem. From the above formula we read off the metric coefficients g_{ij}. These are

$$(g_{ij}) = \begin{pmatrix} \frac{1}{a^2y^2} & 0 \\ 0 & \frac{1}{a^2y^2} \end{pmatrix}$$

We will also need the inverse matrix

$$(g^{ij}) = (g_{ij})^{-1} = \begin{pmatrix} a^2y^2 & 0 \\ 0 & a^2y^2 \end{pmatrix}$$

Now we can compute the Γ^i_{jk}. This is a little inconvenient, but not that bad. We need the basic formulas

$$\Gamma_{ij|k} = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial u^j} + \frac{\partial g_{jk}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^k}\right)$$

and

$$\Gamma^i_{jk} = g^{im}\Gamma_{jk|m}$$

From these we get, remembering that $\Gamma^i_{jk} = \Gamma^i_{kj}$,

$$\begin{aligned}
\Gamma_{11|1} &= \frac{1}{2}\frac{\partial g_{11}}{\partial u^1} = 0 \\
\Gamma_{11|2} &= \frac{1}{2}\Big(\frac{\partial g_{12}}{\partial u^1} + \frac{\partial g_{12}}{\partial u^1} - \frac{\partial g_{11}}{\partial u^2}\Big) = \frac{1}{a^2y^3} \\
\Gamma_{12|1} &= \frac{1}{2}\Big(\frac{\partial g_{11}}{\partial u^2} + \frac{\partial g_{21}}{\partial u^1} - \frac{\partial g_{12}}{\partial u^1}\Big) = -\frac{1}{a^2y^3} \\
\Gamma_{12|2} &= \frac{1}{2}\Big(\frac{\partial g_{12}}{\partial u^2} + \frac{\partial g_{22}}{\partial u^1} - \frac{\partial g_{12}}{\partial u^2}\Big) = 0 \\
\Gamma_{22|1} &= \frac{1}{2}\Big(\frac{\partial g_{21}}{\partial u^2} + \frac{\partial g_{21}}{\partial u^2} - \frac{\partial g_{22}}{\partial u^1}\Big) = 0 \\
\Gamma_{22|2} &= \frac{1}{2}\frac{\partial g_{22}}{\partial u^2} = -\frac{1}{a^2y^3}
\end{aligned}$$

Then

$$\begin{aligned}
\Gamma^1_{11} &= g^{1m}\Gamma_{11|m} = 0 \\
\Gamma^2_{11} &= g^{2m}\Gamma_{11|m} = a^2y^2 \cdot \frac{1}{a^2y^3} = \frac{1}{y} \\
\Gamma^1_{12} &= g^{1m}\Gamma_{12|m} = a^2y^2 \cdot \frac{-1}{a^2y^3} = \frac{-1}{y} \\
\Gamma^2_{12} &= g^{2m}\Gamma_{12|m} = 0 \\
\Gamma^1_{22} &= g^{1m}\Gamma_{22|m} = 0 \\
\Gamma^2_{22} &= g^{2m}\Gamma_{22|m} = a^2y^2 \cdot \frac{-1}{a^2y^3} = \frac{-1}{y}
\end{aligned}$$

For future purposes we place these in matrices:

$$(\Gamma^i_{j1}) = \begin{pmatrix} \Gamma^1_{11} & \Gamma^1_{21} \\ \Gamma^2_{11} & \Gamma^2_{21} \end{pmatrix} = \begin{pmatrix} 0 & \frac{-1}{y} \\ \frac{1}{y} & 0 \end{pmatrix}$$

and

$$(\Gamma^i_{j2}) = \begin{pmatrix} \Gamma^1_{12} & \Gamma^1_{22} \\ \Gamma^2_{12} & \Gamma^2_{22} \end{pmatrix} = \begin{pmatrix} \frac{-1}{y} & 0 \\ 0 & \frac{-1}{y} \end{pmatrix}$$

Now we want to put this information into differential form form; the connection one form is

$$\omega_j{}^i = \Gamma^i_{jk}\, du^k$$

which we can put in matrix form as

$$(\omega_j{}^i) = \begin{pmatrix} 0 & \frac{-1}{y} \\ \frac{1}{y} & 0 \end{pmatrix} du^1 + \begin{pmatrix} \frac{-1}{y} & 0 \\ 0 & \frac{-1}{y} \end{pmatrix} du^2 = \begin{pmatrix} \frac{-1}{y}dy & \frac{-1}{y}dx \\ \frac{1}{y}dx & \frac{-1}{y}dy \end{pmatrix}$$

Our next job is the Riemann Curvature Tensor defined as

$$R_i{}^j{}_{k\ell} = \frac{\partial \Gamma^j_{i\ell}}{\partial u^k} - \frac{\partial \Gamma^j_{ik}}{\partial u^\ell} + \Gamma^j_{mk}\Gamma^m_{i\ell} - \Gamma^j_{m\ell}\Gamma^m_{ik}$$

We will compute one of these by hand and then go over to more efficient methods.

$$\begin{aligned} R_1{}^2{}_{12} &= \frac{\partial \Gamma^2_{12}}{\partial u^1} - \frac{\partial \Gamma^2_{11}}{\partial u^2} + \Gamma^2_{11}\Gamma^1_{12} - \Gamma^2_{12}\Gamma^1_{11} + \Gamma^2_{21}\Gamma^2_{12} - \Gamma^2_{22}\Gamma^2_{11} \\ &= -\frac{\partial \Gamma^2_{11}}{\partial u^2} + \Gamma^2_{11}\Gamma^1_{12} - \Gamma^2_{22}\Gamma^1_{11} \qquad \text{(omitting 0 terms)} \\ &= -\frac{\partial}{\partial y}(\frac{1}{y}) + (\frac{1}{y})(\frac{-1}{y}) - (\frac{-1}{y})(\frac{1}{y}) \\ &= \frac{1}{y^2} - \frac{1}{y^2} + \frac{1}{y^2} \\ &= \frac{1}{y^2} \end{aligned}$$

By coincidence, the formula for Gaussian curvature happens to use $R_1{}^2{}_{12}$ and in fact is

$$K = -\frac{g_{2m}R_1{}^m{}_{12}}{\det(g_{ij})} = -\frac{g_{22}R_1{}^2{}_{12}}{\det(g_{ij})} = -\frac{1}{a^2y^2} \cdot a^4y^4 \cdot \frac{1}{y^2} = -a^2$$

Now we will present the Riemann Curvature tensor in terms of differential forms. We assume a nodding acquaintence with differential forms. The important part to to remember is that $dx \wedge dy = -dy \wedge dx$ and that the proper order is $dx \wedge dy$ and that $dx \wedge dx = dy \wedge dy = 0$.

We have already defined the connection 1-forms as

$$\omega_j{}^i = \Gamma^i_{jk}\, du^k$$

and the Curvature 2-forms are then defined by

$$\Omega = d\omega + \omega \wedge \omega$$

or in less impenetrable language

$$\Omega_j{}^i = d\omega_j{}^i + \omega_k{}^i \wedge \omega_j{}^k$$

We have already manufactured $(\omega_j{}^k)$ above, so we need

$$d(\omega_j{}^i) = d\begin{pmatrix} -\frac{1}{y}dy & -\frac{1}{y}dx \\ \frac{1}{y}dx & -\frac{1}{y}dy \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{y^2}dx \wedge dy \\ \frac{1}{y^2}dx \wedge dy & 0 \end{pmatrix}$$

and

$$(\omega_k{}^i) \wedge (\omega_j{}^k) = \begin{pmatrix} -\frac{1}{y}dy & -\frac{1}{y}dx \\ \frac{1}{y}dx & -\frac{1}{y}dy \end{pmatrix} \wedge \begin{pmatrix} -\frac{1}{y}dy & -\frac{1}{y}dx \\ \frac{1}{y}dx & -\frac{1}{y}dy \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Hence we have the Curvature 2-form $\Omega = d\omega + \omega \wedge \omega$ explicitly as

$$(\Omega_j{}^i) = \begin{pmatrix} 0 & -\frac{1}{y^2}dx \wedge dy \\ \frac{1}{y^2}dx \wedge dy & 0 \end{pmatrix}$$

This tells us that

$$\begin{aligned} \Omega_2{}^1 &= -\frac{1}{y^2}dx \wedge dy \\ \Omega_1{}^2 &= \frac{1}{y^2}dx \wedge dy \end{aligned}$$

and thus that

$$\begin{aligned} R_2{}^1{}_{12} &= -\frac{1}{y^2} \\ R_1{}^2{}_{12} &= \frac{1}{y^2} \end{aligned}$$

This gives us all the coefficients of the Riemann Curvature Tensor if we remember that $R_j{}^i{}_{kl} = -R_j{}^i{}_{lk}$ Note that the result matches what we found previously by cruder methods. Note also the interesting absence of the parameter a, which suggests that $R_i{}^j{}_{kl}$ is really a sort of relative curvature. If we look back at the formula for the Gaussian Curvature we see that a enters only through the (g_{ij}), which provides the absolute scale.

5. FIRST EXAMPLES

The formula

$$ds^2 = \frac{dx^2 + dy^2}{a^2\, y^2}$$

is not very convenient for direct calculation, and part of the fun of Lobachevski geometry is finding clever ways to get around this. However, in a few cases one can directly use the formula to calculate lengths.

The first example is calculating lengths up the y-axis. Since traveling up the y-axis has a dx of 0, the formula becomes

$$\begin{aligned} d(iy_1, iy_2) &= \int_{iy_1}^{iy_2} \frac{dy}{ay} \\ &= \frac{1}{a}\int_{iy_1}^{iy_2} \frac{dy}{y} \\ &= \frac{1}{a}\big(\ln(iy_2) - \ln(iy_2)\big) \\ &= \frac{1}{a}\ln(\frac{iy_2}{iy_1}) \\ &= \frac{1}{a}\ln(\frac{y_2}{y_1}) \end{aligned}$$

which is the formula we want.

We notice now a very important point. Since this distance formula is in terms of the quotient of y_2 and y_1, it is insensitive to a similarity transformation, or homothety, $z \to kz$ where $k > 0$. A moment's reflection shows that, more generally, ds is insensitive to homotheties. This is an important property of the UHP model, and will be used extensively. It is only a special case of our later results, but it is important because of its intuitive directness.

Next we will compute distance along an equidistant. In Fig 1 the vertical line is a Lobachevski straight line as is the Euclidean circle perpendicular to the x-axis. The oblique (Euclidean) line meeting the vertical line at the origin is an equidistant, as is the arc of the second circle which does *not* meet the x-axis at right angles. An equidistant is a Lobachevski curve having a fixed distance from a straight line. In both cases in Fig 1, the arcs P_1Q_1 and P_2Q_2 are Lobachevski straight lines of equal lengths. We will find these lengths later and show their equality. For now we are interested in the the distance from Q_1 to Q_2 on the oblique Euclidean line which makes an angle θ with the vertical, the angle counting as positive to the right and negative to the left, as is the case in Fig 1. The equation of the equidistant is $x = y\tan(\theta)$ The points are $Q_1 = z_1 = x_1 + iy_1$ and $Q_2 = z_2 = x_2 + iy_2$. Then

$$d(Q_1, Q_2) = \int_{z_1}^{z_2} ds = \int_{x_1+iy_1}^{x_2+iy_2} \frac{\sqrt{dx^2 + dy^2}}{ay}$$

$$
\begin{aligned}
&= \int_{iy_1}^{iy_2} \frac{\sqrt{\tan^2\theta + 1}dy}{ay} \\
&= \frac{\sqrt{\tan^2\theta + 1}}{a} \int_{iy_1}^{iy_2} \frac{dy}{y} \\
&= \frac{\sec\theta}{a} \ln \frac{y_2}{y_1}
\end{aligned}
$$

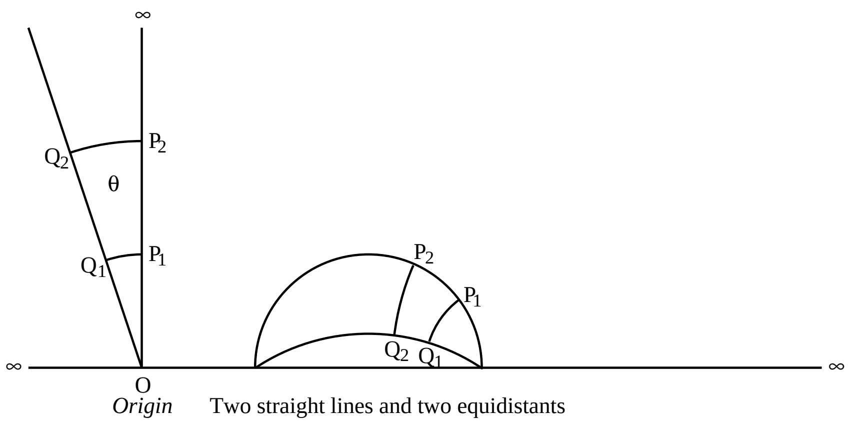

Figure 1: Some straight lines and equidistants

Equidistants have no analogy in Euclidean geometry because there an equidistant to a stright line is a straight line, and a pair of equidistant straight lines will force the geometry to be Euclidean locally. For example, two circles around a circular cylinder are equidistant, and these are straight lines (geodesics) on the cylinder, so the cylinder is locally Euclidean. Equidistants and the soon to be introduced horocycles make Lobachevski geometry much more amusing than Euclidean geometry.

Straight lines form a 2-parameter family of curves in Lobachevski space. We can parametrize them by center on $\mathbb{R}$ or at ∞ and radius $r \in (0, \infty)$ so a possible parameter space for the straight lines is

$$\mathbb{P}^1(\mathbb{R}) \times \mathbb{R}$$

Equidistants are determined by $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \equiv \mathbb{R}$ so they form a 3-parameter with possible parameter space

$$\mathbb{P}^1(\mathbb{R}) \times \mathbb{R} \times \mathbb{R}$$

Before introducing horocyles we note (and will later prove) that Euclidean circles and Lobacevski circles in the upper half plane are identical as point sets, but it is very important to realize that the Euclidean center does not coincide with the Lobachevski center, which is always lower than the Euclidean center. For example, a Euclidean circle with center $2i$ and radius 1 goes throught the points i and $3i$. The point equidistant in Lobachevski distance from these two is $\sqrt{3}i$ which is lower than $2i$. Indeed

$$d(i, \sqrt{3}\,i) = \ln \frac{\sqrt{3}}{1} = \frac{1}{2}\ln 3$$

and

$$d(\sqrt{3}\,i, 3i) = \ln \frac{3}{\sqrt{3}} = \ln \sqrt{3} = \frac{1}{2}\ln 3$$

We note that given a circle with center on the y-axis and going through the two points $y_1\, i$ and $y_2\, i$ the Euclidean center is the arithmetic mean and the Lobachevski center is the geometric mean of y_1 and y_2.

The next Lobachevski curve which has no Euclidean analogy is the horocycle. Horocycles result in two (merely cosmetically) different ways. If the lowest point on a Lobachevski circle is fixed and the center moves vertically upwards the limit will be a horizontal line which is the first form of the horocycle. Similarly if the highest point on a Lobachevski circle is fixed and the center moved vertically toward the x-axis the limit will be a horocycle which appears in the UHP model as a circle tangent to the x-axis. Thus horocycles, very like straight lines, form a 2-parameter family which can be parametrized by

$$\mathbb{P}^1(\mathbb{R}) \times \mathbb{R}$$

We will later show that all horocycles are congruent. As we would guess, it is

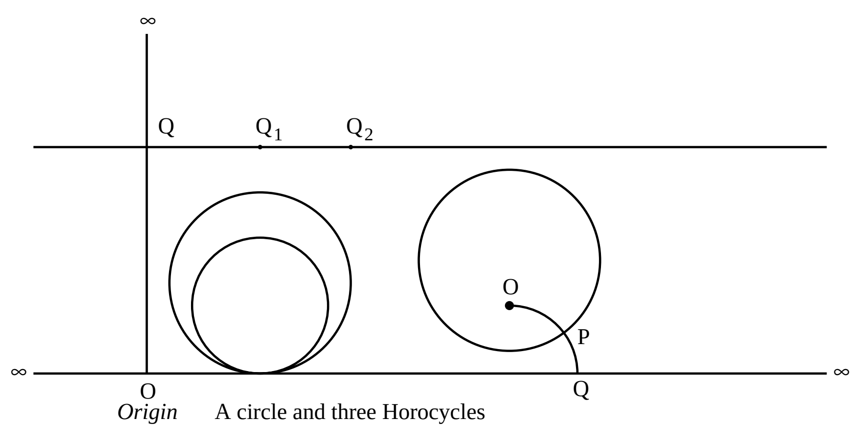

Figure 2: Some horocycles and a circle

easy to find the length along the horizontal line horocycle. We set things up as in the figure with $Q = yi$, $Q_1 = x_1 + yi$, $Q_2 = x_2 + yi$ and $\tan(\theta_1) = \frac{x_1}{y}$. Then we have

$$\begin{aligned}
d(Q, Q_1) &= \int_{iy}^{x_1+iy} ds = \int_{iy}^{x_1+iy} \frac{dx}{ay} \\
&= \frac{1}{ay} \int_{iy}^{y\tan(\theta_1)+iy} dx = \frac{1}{ay}(y\tan(\theta_1) + iy - iy) \\
&= \frac{1}{a}\tan(\theta_1)
\end{aligned}$$

thus we see that the Length depends only on the subtended angles

$$d(Q_1, Q_2) = \frac{1}{a}(\tan(\theta_2) - \tan(\theta_1))$$

and not on the height y, as we would expect given the invariance of ds under homothety.

Since neither equidistants nor horocycles are straight lines they must be curved lines and hence have curvature. We mean at this point to investigate the curvature of horocyles and staight lines, which uses slightly heavier equipment than we have used up to this point. The casual reader might well wish to skip to the next section or skim the rest of this section and read it in detail when she needs it. The calculation is not too bad for horocycles but a little annoying for equidistants.

Since all horocycles are congruent, the curvature calculation should come out the same for them all, as we will show when we have more equipment. For now we will show that the horizontal line horocycle has non-zero constant curvature independent of y.

The concept we need here is the geodesic curvature. In order to calculate with the following formula the curve parameter must be the arc length s, which fortunately in the cases we need is fairly simple. The formula is

$$\kappa_g = \varepsilon_{li} u'^l (u''^i + \Gamma^i_{jk} u'^j u'^k)$$

where the prime indicates differentiation with respect to arc length and ε_{li} is defined by

$$\begin{aligned}
\varepsilon_{12} &= \sqrt{g} = \sqrt{\det(g_{ij})} \\
\varepsilon_{21} &= -\sqrt{g} \\
\varepsilon_{11} &= \varepsilon_{22} = 0
\end{aligned}$$

and

$$u^1 = x, \qquad u^2 = y$$

and for the horocycle

$$u'^2 = u''^2 = 0$$

Hence, putting in only the non-zero terms,

$$\kappa_g = \varepsilon_{12} u'^1 (u''^2 + \Gamma^2_{11} u'^1 u'^1) + \varepsilon_{21} u'^2 (u''^1 + \Gamma^1_{11} u'^1 u'^1)$$

The tricky part is that this formula assumes parametrization by arc length. But we know $s = (1/a)\tan(\theta)$ so $s = x/(ay)$ and so $u^1 = x = ays$ and so $\dot{u}^1 = ay$ and $\ddot{u}^1 = 0$. Hence κ_g comes down to

$$
\begin{aligned}
\kappa_g &= \varepsilon_{12} ay(0 + \Gamma^2_{11}(ay)(ay)) + 0 \\
&= \sqrt{\frac{1}{a^4y^4}}\,\frac{1}{y}\,(ay)^3 = \frac{1}{a^2y^2}\,\frac{1}{y}\,a^3y^3 \\
&= a = \sqrt{-K}
\end{aligned}
$$

where K is the Gaussian Curvature (which is negative). As we predicted the geodesic cuvature is the same for every point on every horizontal horocycle. We will later show that this is true for horocyles in general.

For equidistants things are rather more complicated. We will use the oblique Euclidean line characterized by the angle θ. Since the value $\theta = 0$ gives an actual straight line we would hope that $\kappa_g(0) = 0$ if, as we suspect, κ_g turns out to be a function of θ that increases with θ. So it is now time for the computation. For equidistants (taking $Q_1 = i$)

$$
u^1 = x = y\tan(\theta) \qquad u^2 = y \qquad s = \frac{\sec(\theta)}{a}\ln(\frac{y}{1})
$$

Thus

$$
\begin{aligned}
u^2 &= y = e^{\frac{as}{\sec(\theta)}} = e^{as\cos(\theta)} \\
u^1 &= y\tan(\theta) = \tan(\theta)e^{as\cos(\theta)} \\
\dot{u}^2 &= a\cos(\theta)e^{as\cos(\theta)} \\
\dot{u}^1 &= \tan(\theta)\dot{u}^2 = a\tan(\theta)\cos(\theta)e^{as\cos(\theta)} = a\sin(\theta)e^{as\cos(\theta)} \\
\ddot{u}^2 &= a^2\cos^2(\theta)e^{as\cos(\theta)} \\
\ddot{u}^1 &= \tan(\theta)\ddot{u}^2 = a^2\sin(\theta)\cos(\theta)e^{as\cos(\theta)}
\end{aligned}
$$

Putting all this into the formula for geodesic curvature (and writing down only the non-zero terms) we have

$$
\begin{aligned}
\kappa_g &= \varepsilon_{li}\dot{u}^l(\ddot{u}^i + \Gamma^i_{jk}\dot{u}^i\dot{u}^j) \\
&= \varepsilon_{12}\dot{u}^1(\ddot{u}^2 + \Gamma^2_{11}\dot{u}^1\dot{u}^1 + \Gamma^2_{22}\dot{u}^2\dot{u}^2) + \varepsilon_{21}\dot{u}^2(\ddot{u}^1 + \Gamma^1_{12}\dot{u}^1\dot{u}^2 + \Gamma^1_{21}\dot{u}^2\dot{u}^1) \\
&= \sqrt{g}e^{2as\cos(\theta)}a^3\{\dot{u}^1\ddot{u}^2 - \dot{u}^2\ddot{u}^2\} \\
&+ \sqrt{g}e^{3as\cos(\theta)}a^3\{\sin(\theta)(\frac{1}{y}\sin^2(\theta) - \frac{1}{y}\cos^2(\theta)) \\
&- \cos(\theta)(-\frac{1}{y}\sin(\theta)\cos(\theta) - \frac{1}{y}\cos(\theta)\sin(\theta))\} \\
&= \sqrt{g}e^{2as\cos(\theta)}a^3\{\dot{u}^1\tan(\theta)\ddot{u}^1 - \tan(\theta)\dot{u}^1\ddot{u}^1\} \\
&+ \sqrt{g}y^3a^3\frac{1}{y}\{\sin(\theta)\sin^2(\theta) - \sin(\theta)\cos^2(\theta) + \sin(\theta)\cos^2(\theta) + \cos^2(\theta)\sin(\theta)\} \\
&= 0 + \frac{1}{a^2y^2}a^3y^2\sin(\theta) \\
&= a\sin(\theta)
\end{aligned}
$$

So we have shown that indeed κ_g is a function $\kappa_g(\theta)$ of θ. Another important point is that, as predicted, $\kappa_g(0) = 0$ and when the equidistant is actually a straight line (the y-axis) the geodesic curvature is indeed 0, at least for the vertical kind of straight lines. We will soon see we can carry this over to the other straight lines also. This was a critical point to verify, since it means our choice of metric is consistent with our choice of what should count as a straight line.

6. MÖBIUS TRANSFORMATIONS AND THE CROSS RATIO

To go further with the metric properties of the UHP we will need some additional equipment. Recall that

$$Tz = \frac{az+b}{cz+d} \qquad\qquad ad-bc \neq 0$$

is a fractional linear or Möbius transformation which maps the Riemann Sphere $\mathbb{P}(\mathbb{C})$ onto itself in a bijective manner. ($\infty \to a/c$, $-d/c \to \infty$, with obvious modifications if $c = 0$). Möbius transformations have a great many interesting properties but the one that will be most critical for us initially is the

Theorem A Möbius transformation takes lircles to lircles.

The proof of this theorem uses methods somewhat different from those of this module and so has been put in its own module entitled *Möbius transformations and lircles.*

It is important that Möbius transformations are closely connected to matrices but as we will not make any use of this connection here we will not go into it at this time.

We need the inverse of a Möbius transformation which is

$$T^{-1}z = \frac{dz-b}{-cz+a}$$

Next we deal with the cross ratio of four points. It is defined by

$$D(z_1,z_2,z_3,z_4) = \frac{\frac{z_1-z_3}{z_1-z_4}}{\frac{z_2-z_3}{z_2-z_4}} = \frac{(z_1-z_3)}{(z_1-z_4)}\frac{(z_2-z_4)}{(z_2-z_3)}$$

The cross ratio was originally developed as an invariant of projective transformations of the extended Euclidian line $\mathbb{P}^1(\mathbb{R})$. Since the Riemann Sphere $\mathbb{P}^1(\mathbb{C})$ is a close analog of the extended line, it is not surprising the cross ratio will have similar properties for it.

There are difficulties if too many of the points z_1, z_2, z_3, z_4 are identical. We will work with the case where at least three of them are distinct, in which case there is no difficulty. For example

$$D(z_1,z_2,z_2,z_4) = \frac{(z_1-z_2)}{(z_1-z_4)}\frac{(z_2-z_4)}{(z_2-z_2)} = \infty$$

This ∞ is the $\infty \in \mathbb{P}^2(\mathbb{C})$ and we will identify it with the ∞ atop the y-axis in the UHP when convenient.

Next we have a few useful Lemmas and a critical theorem.

Lemma If T takes 1, 0, ∞ to 1, 0, ∞ in that order then $Tz = z$

Proof sketch: $T(\infty) = \infty \Rightarrow c = 0$, $T(0) = 0 \Rightarrow b = 0$ and $T(1) = 1 \Rightarrow a = d$

Lemma If z_2, z_3, z_4 are distinct points in $\mathbb{C}$, then there is a unique Möbius transformation T which takes z_2, z_3, z_4 to 1, 0, ∞ in that order.

Proof: Tz is defined if no $z_i = \infty$ by

$$Tz = \frac{z - z_3}{z - z_4} \frac{z_2 - z_4}{z_2 - z_3}$$

If $z_i = \infty$ for some i then one finds the correct form by taking limits:

$$Tz = \frac{z - z_3}{z_2 - z_3} \frac{z_2 - z_4}{z - z_4} \to \frac{z_2 - z_4}{z - z_4} \qquad \text{as } z_3 \to \infty$$

Uniqueness follows from the previous Lemma.

Lemma $D(z_1, z_2, z_3, z_4)$ gives the value $T(z_1)$ where T is the Möbius transformation defined in the previous lemma

Proof:

$$Tz_1 = \frac{z_1 - z_3}{z_1 - z_4} \frac{z_2 - z_4}{z_2 - z_3} = D(z_1, z_2, z_3, z_4)$$

Theorem D is invariant under Möbius transformations;

$$D(Tz_1, Tz_2, Tz_3, Tz_4) = D(z_1, z_2, z_3, z_4)$$

Proof: Let $Sz = D(z, z_2, z_3, z_4)$. Then $D(Tz_1, Tz_2, Tz_3, Tz_4) =$ the value of $U(Tz_1)$ where U is defined as the transformation satifying

$$UT(z_2) = 1, \quad UT(z_2) = 0, \quad UT(z_2) = \infty$$

But

$$S(z_2) = 1, \quad S(z_2) = 0, \quad S(z_2) = \infty$$

By uniqueness $UT = S$. Thus

$$D(Tz_1, Tz_2, Tz_3, Tz_4) = UTz_1 = Sz_1 = D(z_1, z_2, z_3, z_4)$$

This gives us a method of finding the Möbius transformation that takes z_2, z_3, z_4 into w_2, w_3, w_4. We just solve

$$D(w, w_2, w_3, w_4) = D(z, z_2, z_3, z_4)$$

for w in terms of z.

We can now prove the important

Theorem The cross ratio of four distinct points is real if and only if the four distinct points lie on a lircle.

Proof: Let the four points be z_1,z_2, z_3, and z_4, and let T be the Möbius transfomation that takes

$$Tz_2 = 1, \qquad Tz_3 = 0, \qquad Tz_4 = \infty$$

Let C be the unique lircle though z_2, z_3, and z_4. Since a Möbius transformation takes lircles to lircles, $T[C] = \mathbb{R}$.
$\Leftarrow$: Let $z_1 \in C$. Then $Tz_1 \in \mathbb{R}$. But then $D(z_1, z_2, z_3, z_4) = D(Tz_1, 1, 0, \infty) = Tz_1$ which is real.
$\Rightarrow$: Suppose $D(z_1, z_2, z_3, z_4)$ is real. Then

$$\begin{aligned} Tz_1 &= D(Tz_1, 1, 0, \infty) = D(Tz_1, Tz_2, Tz_3, Tz_4) \\ &= D(z_1, z_2, z_3, z_4) \end{aligned}$$

so $Tz_1 \in \mathbb{R} = T[C]$. Hence $z_1 \in C$ since T is bijective.

It is useful to have a few facts about permuting the order of terms in the cross ratio. It is easiest to figure these out using $D(x, 1, 0, \infty) = x$ and then tranfer the knowledge to $D(z_1, z_2, z_3, z_4)$ by a Möbius transformation. There are $24 = 4!$ permutations of 4 elements. The cross ratio has, in general, 6 different values each of which corresponds to 4 permutations. In the following table x may be complex. The D in $D(x, 1, 0, \infty)$ is omitted in the table to minimize clutter.

x	$(x, 1, 0, \infty)$	$(1, x, \infty, 0)$	$(0, \infty, x, 1)$	$(\infty, 0, 1, x)$
$\frac{1}{x}$	$(1, x, 0, \infty)$	$(x, 1, \infty, 0)$	$(0, \infty, 1, x)$	$(\infty, 0, x, 1)$
$\frac{1}{1-x}$	$(0, x, 1, \infty)$	$(x, 0, \infty, 1)$	$(1, \infty, 0, x)$	$(\infty, 1, x, 0)$
$1 - x$	$(x, 0, 1, \infty)$	$(0, x, \infty, 1)$	$(1, \infty, x, 0)$	$(\infty, 1, 0, x)$
$\frac{x-1}{x}$	$(1, 0, x, \infty)$	$(0, 1, \infty, x)$	$(x, \infty, 1, 0)$	$(\infty, x, 0, 1)$
$\frac{x}{x-1}$	$(0, 1, x, \infty)$	$(1, 0, \infty, x)$	$(x, \infty, 0, 1)$	$(\infty, x, 1, 0)$

By mapping $Tz_1 = x$, $Tz_2 = 1$, $Tz_3 = 0$, $Tz_4 = \infty$ and using invarience of the cross ratio we see, for example, that

$$\begin{aligned} \frac{1}{D(z_1, z_2, z_3, z_4)} &= D(z_2, z_1, z_3, z_4) = D(z_1, z_2, z_4, z_3) \\ &= D(z_3, z_4, z_2, z_1) = D(z_4, z_3, z_1, z_2) \end{aligned}$$

The six functions in the table form a group under composition which is a representation of S_4 but we need not go into this here.

Now a few words about orientation. We have

$$D(x, 1, 0, \infty) = \frac{\frac{x-0}{x-\infty}}{\frac{1-0}{1-\infty}} = \frac{x}{1}\frac{1-\infty}{x-\infty} = x$$

and

$$D(1, x, 0, \infty) = \frac{1}{x}$$

If x is real then these cross ratios are positive when x is postive. Because a holomorphic function preserves orientation, we can map a lircle with the points z_1, z_2, z_3, z_4 going around the lircle clockwise to x, 0, 1, ∞ with real x which must satisfy $x < 0$ to preserve the orientation. Thus

$$D(z_1, z_2, z_3, z_4) = D(x, 0, 1, \infty) = 1 - x > 0$$

In the other direction

$$D(z_4, z_3, z_2, z_1) = D(z_2, z_1, z_4, z_3) = D(z_1, z_2, z_3, z_4) > 0$$

Hence the cross ratio is positive if the points z_1, z_2, z_3, z_4 on the circle are arranged in order going around the circle in either direction, which we will call coherant ordering. What happens if we excange the middle pair?

$$D(z_1, z_3, z_2, z_4) = D(x, 1, 0, \infty) = x < 0$$

If we exchange the middle two points in a coherant set the resulting cross ratio is negative.

We note that the T taking a coherant ordering to x, 0,1, ∞ must take the exterior and the interior of the circle to the two half planes determined by $\mathbb{R}$. If the ordering is clockwise then the interior goes to the lower half plane by continuity, and similarly if the ordering is counterclockwise the interior goes to the upper half plane.

Example Let the lircle be the unit circle and $z_2 = -i$, $z_3 = 1$, and $z_4 = i$. Then

$$Tz = \frac{1}{i}\,\frac{z+i}{z-i}$$

Indeed we have

$$\begin{aligned} T(-i) &= 0 \\ T(1) &= \frac{1}{i}\,\frac{1+i}{1-i} = \frac{1+i}{i+1} = 1 \\ T(i) &= \infty \end{aligned}$$

Our prediction is that the interior of the unit circle will be carried to the upper half plane and indeed

$$T(0) = \frac{1}{i}\,\frac{i}{-i} = i \in \mathrm{UHP}$$

One further point; we will be interested in the particular combinations

$$D(z_2, z_3, z_1, z_4) \qquad D(z_3, z_2, z_4, z_1)$$

for coherantly placed z_1, z_2, z_3, z_4 on a lircle. Mapping the points to x, 0, 1, ∞ as before (with $x < 0$) we have

$$\begin{aligned} D(z_2, z_3, z_1, z_4) &= D(0, 1, x, \infty) = \frac{x}{x-1} > 0 \\ D(z_3, z_2, z_4, z_1) &= D(1, 0, x, \infty) = \frac{x-1}{x} > 0 \end{aligned}$$

So the two cross ratios of interest are postive if the points are coherantly arranged.

7. MOTIONS OF THE UHP AND DISTANCE

We now wish to investigate distance and motions in the upper half plane (UHP) model of Lobachevski geometry, which is quite a bit less straightforward than in the Euclidean plane.

An *isometry* is a function $f : X \to Y$ where X and Y are metric spaces and f preserves distance:

$$d(f(x), f(y)) = d(x, y)$$

If X and Y have orientations then f can be orientation preserving or orientation reversing (assuming X is connected). Isometries are injective but need not be surjective.

If $Y = X$ then a surjective isometry is often called a motion, and if it preserves orientation it is called an orientation preserving motion or a *proper* motion. For example, in Euclidean Space two figures are congruent if and only if there is a motion which takes one to the other. Reflections in Euclidean space are examples of motions which are *not* proper motions; they reverse orientation. We will be concerned in this section with the proper motions of Lobachevski space, which turn out to be certain Möbius transformations.

It is important to realize that there are two rather different formulas for the distance from z_1 to z_2 in the Lobachevski plane. Let l be the lircle perpendicular to the x-axis through z_1 and z_2 with z_0 and z_∞ being the points where the lircle hits $\mathbb{R} \cup \{\infty\}$ and arranged coherantly; that is z_0, z_1, z_2, and z_∞ follow one another along the lircle in that order. Then

$$d(z_1, z_2) = \frac{1}{a} \ln D(z_2, z_1, z_0, z_\infty)$$

where $D(z_2, z_1, z_0, z_\infty)$ is the cross ratio of the four points as discussed in the previous section:

$$D(z_2, z_1, z_0, z_\infty) = \frac{z_2 - z_0}{z_2 - z_\infty} \frac{z_1 - z_\infty}{z_1 - z_0}$$

A result in the previous section guarantees that the cross ratio will be real and positive in this case, but this will also be obvious from the derivation, which is easy.

The second formula is

$$d(z_1, z_2) = \frac{2}{a} \operatorname{argsinh} \frac{|z_2 - z_1|}{2\sqrt{\Im z_1}\sqrt{\Im z_2}}$$

which is more difficult to derive although I present a reasonably plausible path. It is not trivial to get from the first formula to the second formula, although it is probably possible with an orgy of trig and hyperbolic functions, though I have not succeeded in doing it.

Both formulas are important for future use and will be derived after some preliminary material on motions.

A third formula

$$d(z_1, z_2) = \frac{2}{a} \operatorname{argtanh} \frac{|z_2 - z_1|}{|\overline{z}_2 - z_1|}$$

will be derived in Book 2. It is a variant of the second formula.

Our first important job is to identify those Möbius transformations that take the upper half plane (UHP) to itself. This is easy and will eventually provide us with a large supply of motions of the Lobachevski plane.

The topological closure of the UHP (regarded as a subset of the Riemann sphere) is

$$\overline{U} = \text{UHP } \cup \mathbb{R} \cup \{\infty\}$$

We then have

Theorem A Möbius transformation T is a bijection of $\overline{U}$ if and only if T has a representation in which

$$a, b, c, d \in \mathbb{R} \quad \text{and} \quad \det(\begin{matrix} a & b \\ c & d \end{matrix}) > 0$$

Proof: (sketch) Because T must take $\partial U = \mathbb{R}$ into $\partial U = \mathbb{R}$, it is fairly obvious that we can restrict our attention to $a, b, c, d \in \mathbb{R}$. Let $z \in$ UHP so $\Im z > 0$. Then

$$\begin{aligned} \Im T z &= \frac{1}{2i}\Big(\frac{az+b}{cz+d} - \frac{a\overline{z}+b}{c\overline{z}+d}\Big) \\ &= \frac{1}{2i}\frac{ad(z-\overline{z}) - bc(z-\overline{z})}{(cz+d)(c\overline{z}+d)} \\ &= \frac{ad-bc}{|cz+d|^2}\Im z \end{aligned}$$

Hence

$$\Im T z > 0,\, \Im z > 0 \Rightarrow \det(\begin{matrix} a & b \\ c & d \end{matrix}) = ad - bc > 0$$

and

$$\Im z > 0,\, \det(\begin{matrix} a & b \\ c & d \end{matrix}) > 0 \Rightarrow \Im T z > 0$$

Bijectivity follows from the bijectivity of T on the Riemann sphere.

Recall that the metric for the UHP model of Lobachevski space is

$$\begin{aligned} ds^2 &= \frac{dx^2 + dy^2}{a^2 y^2} \\ &= \frac{(dx + idy)(dx - idy)}{a^2\,(\frac{z-\overline{z}}{2i})^2} \\ &= \frac{-4dz\, d\overline{z}}{a^2(z-\overline{z})^2} \end{aligned}$$

We wish to show that $w = Tz$ with $a, b, c, d \in \mathbb{R}$ and $ad - bc > 0$ is an isometry of the UHP. To do this it suffices to show that

$$\frac{-4dw\ d\overline{w}}{a^2(w - \overline{w})^2} = ds^2 = \frac{-4dz\ d\overline{z}}{a^2(z - \overline{z})^2}$$

Since distance along a curve is determined by integrating ds, this will show that distance along a curve C from z_1 to z_2 is the same as distance along $T[C]$ from $T(z_1)$ to $T(z_2)$, which is enough to guarantee isometry.

However, we have other ends in mind besides derivation of the formula for isometry, which is why the following is not as straightforward as one might expect.

The primary formula for this material is $w_2 - w_1$ in terms of $z_2 - z_1$:

$$\begin{aligned}
w_2 - w_1 &= \frac{az_2 + b}{cz_2 + d} - \frac{az_1 + b}{cz_1 + d} \\
&= \frac{acz_1z_2 + adz_2 + bcz + bd - acz_1z_2 - bcz_2 - adz_1 - bd}{(cz_1 + d)(cz_2 + d)} \\
&= \frac{(ad - bc)(z_2 - z_1)}{(cz_1 + d)(cz_2 + d)} \\
&= \frac{\Delta(z_2 - z_1)}{(cz_1 + d)(cz_2 + d)}
\end{aligned}$$

where, as usual, $\Delta = ad - bc$. Just for amusement I note

$$\frac{dw}{dz} = \lim_{z_2 \to z} \frac{w_2 - w}{z_2 - z} = \frac{\Delta}{(cz + d)^2}$$

so

$$dw = \frac{\Delta\, dz}{(cz + d)^2}$$

Because $a, b, c, d \in \mathbb{R}$ we have $\overline{w} = T\overline{z}$. Hence our formula gives

$$w - \overline{w} = \frac{\Delta(z - \overline{z})}{(cz + d)(c\overline{z} + d)}$$

Dividing the previous two formulas

$$\frac{dw}{w - \overline{w}} = \frac{dz}{z - \overline{z}} \frac{c\overline{z} + d}{cz + d},$$

a formula I like although I have found no direct use for it. Conjugating we have

$$\frac{d\overline{w}}{\overline{w} - w} = \frac{d\overline{z}}{\overline{z} - z} \frac{cz + d}{c\overline{z} + d}$$

Multiplying the last two formulas together and then multiplying both sides by $-4/a^2$ gives

$$\frac{-4\, dw\, d\overline{w}}{a^2(w - \overline{w})^2} = \frac{-4\, dz\, d\overline{z}}{a^2(z - \overline{z})^2}$$

which proves

Theorem The Möbius transformation T with $a, b, c, d \in \mathbb{R}$ and $ad - bc > 0$ is a proper motion of the Upper Half Plane.

Proof: Bijectivity follows from T being a Möbius transformation, proper comes from the orientation preserving property of any holomorphic funtion, and isometry was just proved above.

Next we will prove a formula which will later be critical for the second distance formula. Observe the last formula may be rewritten as

$$\frac{|dw|^2}{a^2(\Im w)^2} = \frac{|dz|^2}{a^2(\Im z)^2}$$

Since dz mystically suggests $z_2 - z_1$ (a difference) we conjecture that the quantity

$$\frac{|z_2 - z_1|^2}{a^2(\Im z_1)(\Im z_2)}$$

might prove interesting. Recall we started with

$$w_2 - w_1 = \frac{\Delta\,(z_2 - z_1)}{(cz_1 + d)(cz_2 + d)}$$

so

$$\overline{w_2} - \overline{w_1} = \frac{\Delta\,(\overline{z_2} - \overline{z_1})}{(c\overline{z_1} + d)(c\overline{z_2} + d)}$$

and recall

$$2i\,\Im w_1 = w_1 - \overline{w_1} = \frac{\Delta\,(z_1 - \overline{z_1})}{(cz_1 + d)(c\overline{z_1} + d)} = \frac{\Delta\, 2i(\Im z_1)}{(cz_1 + d)(c\overline{z_1} + d)}$$

and

$$2i\,\Im w_2 = w_2 - \overline{w_2} = \frac{\Delta\,(z_2 - \overline{z_2})}{(cz_2 + d)(c\overline{z_2} + d)} = \frac{\Delta\, 2i(\Im z_2)}{(cz_2 + d)(c\overline{z_2} + d)}$$

Of the last four formulas, cancel the $2i$'s in the last two and then divide the product of the first two formulas by the product of the last two formulas; the Δ's and the denominators will cancel out and we will be left with

$$\frac{(w_2 - w_1)(\overline{w_2} - \overline{w_1})}{(\Im w_1)(\Im w_2)} = \frac{(z_2 - z_1)(\overline{z_2} - \overline{z_1})}{(\Im z_1)(\Im z_2)}$$

It is important for proper appreciation of this result to notice that the cancelled out denominator $(cz_1+d)(c\overline{z_1}+d)(cz_2+d)(c\overline{z_2}+d)$ is formed in different ways in the product of the first two equations and the product of the last two equations, but comes out the same in the end. It is this circumstance that makes it all work. This circumstance occurs from time to time in mathematics and is often important.

The importance of the last equation is that the expression

$$\frac{|z_2 - z_1|^2}{(\Im z_1)(\Im z_2)}$$

is invariant under Möbius transformations, like ds^2. However, one must realize that it is *not* the finite analog of ds^2, which is a little more complicated. We will return to this matter later.

We now turn out attention to the derivation of the first distance formula in UHP. Given points z_1 and z_2 we put a Lobachevski straight line l (which is uniquely determined by the two points) through them and and we determine z_0 and z_∞ as the points where this straight line hits ∂(UHP)$= \mathbb{R} \cup \{\infty\}$ chosen in such a way so that if one travels along the straight line from z_0 toward z_∞ one first encounters z_1 and then z_2. That is, one encounters the points in the order z_0, z_1, z_2, z_∞ on the half-lircle. We now form a Möbius transformation T by requiring that

$$Tz_0 = 0, \qquad Tz_1 = i, \qquad Tz_\infty = \infty$$

In fact, the Möbius transformation T is given (typically) by

$$Tz = i\,\frac{z - z_0}{z - z_\infty}\,\frac{z_1 - z_\infty}{z_1 - z_0}$$

(If l is a vertical Euclidean line one makes the obvious modifications.)

Since the positive y-axis is a geodesic (as we have previously proved) and since T^{-1} takes the y-axis to l and since T^{-1} is a motion (isometery) it follows that l is also a geodesic and that $d(z_1, z_2) =$ distance along l from z_1 to z_2. But since T is a motion we have

$$\begin{aligned} d(z_1, z_2) &= d(Tz_1, Tz_2) = d(i, iy) \\ &= \frac{1}{a}\,\ln\frac{iy}{i} = \frac{1}{a}\,\ln\left(\frac{1}{i}\,iy\right) \\ &= \frac{1}{a}\,\ln\left(\frac{1}{i}\,Tz_2\right) \\ &= \frac{1}{a}\,\ln\frac{z_2 - z_0}{z_2 - z_\infty}\,\frac{z_1 - z_\infty}{z_1 - z_0} \\ &= \frac{1}{a}\,\ln D(z_2, z_1, z_0, z_\infty) \end{aligned}$$

where $D(z_2, z_1, z_0, z_\infty)$ is the cross ratio. Note the unintuitive order of the z_i's. Thus we have derived the first distance formula fairly easily. Note that if we reverse the roles of z_1 and z_2 we will also reverse the roles of z_0 and z_∞ and so

$$\begin{aligned} d(z_2, z_1) &= \frac{1}{a}\,D(z_1, z_2, z_\infty, z_0) \\ &= \frac{1}{a}\,D(z_2, z_1, z_0, z_\infty) = d(z_1, z_2) \end{aligned}$$

using properties of the cross ratio (which can also be verified directly). Now let z_0, z_1, z_2, z_3, z_∞ go along the Lobacevski straight line in the order given. Then

$$\begin{aligned} D(z_2, z_1, z_0, z_\infty)D(z_3, z_2, z_0, z_\infty) &= \frac{z_2 - z_0}{z_2 - z_\infty}\frac{z_1 - z_\infty}{z_1 - z_0}\frac{z_3 - z_0}{z_3 - z_\infty}\frac{z_2 - z_\infty}{z_2 - z_0} \\ &= \frac{z_3 - z_0}{z_3 - z_\infty}\frac{z_1 - z_\infty}{z_1 - z_0} \\ &= D(z_3, z_1, z_0, z_\infty) \end{aligned}$$

so that, taking logs we have

$$d(z_1, z_2) + d(z_2, z_3) = d(z_1, z_3)$$

Example Now for fun let's do an example. Let $z_1 = -4 + 3i$ and $z_2 = 4 + 3i$. The straight line l is thus the Euclidean semicircle of radius 5 and center at the origin. Hence $z_0 = -5$ and $z_\infty = 5$. Then using our formulas

$$\begin{aligned} d(z_1, z_2) &= \frac{1}{a}\ln\Big(\frac{4+3i+5}{4+3i-5}\frac{-4+3i-5}{-4+3i+5}\Big) = \frac{1}{a}\ln\Big(\frac{9+3i}{-1+3i}\frac{-9+3i}{1+3i}\Big) \\ &= \frac{1}{a}\ln\Big(\frac{-90}{-10}\Big) \approx 2.19722\,\frac{1}{a} \end{aligned}$$

For contrast, let us calculate the distance along the (horizontal) horocyle on which z_1 and z_2 lie. Here $dy = 0$. If h is the segment of the horocyle we have

$$\begin{aligned} \int_h ds &= \int_{-4+3i}^{4+3i}\frac{dx}{ay} = \frac{1}{3a}\int_{-4+3i}^{4+3i} dx \\ &= \frac{1}{3a}\big(4+3i-(-4+3i)\big) = \frac{1}{a}\,\frac{8}{3} \approx 2.666\,\frac{1}{a} \end{aligned}$$

Hence, as expected, it is shorter along the Lobachevski straight line (which is a geodesic) than along the horocycle (which is a *curve* in Lobachevski space). Note the ratio of the two lengths is independent of a, whose role is to set an absolute scale of length.

From the formula

$$d(z_1, z_2) = \frac{1}{a}\ln\frac{z_2 - z_0}{z_2 - z_\infty}\frac{z_1 - z_\infty}{z_1 - z_0}$$

we see that as z_1 approaches z_0 (other points being fixed) along the Lobachevski straight line we have $d(z_1, z_2) \to \infty$. The points on $\mathbb{R} \cup \{\infty\}$ are infinitly far away from all points in the UHP.

For various purposes we would like to derive a formula for the distance from a point z to the positive y-axis. Let the line Oz make an angle θ with the positive y-axis and draw a Euclidean circle with center 0 and radius $|z|$. The Euclidean circle is perpendicular to the positive y-axis and is a Lobachevski straight line.

This might be described as “dropping a perpendicular to the positive y-axis”. Hence the distance sought is $d(i|z|, z)$ and we have

$$\begin{aligned} d(i|z|, z) &= \frac{1}{a}\ln D(z, i|z|, -|z|, |z|) \\ &= \frac{1}{a}\ln\Big(\frac{z+|z|}{z-|z|}\,\frac{i|z|-|z|}{i|z|+|z|}\Big) = \ln\Big(\frac{z+|z|}{z-|z|}\,\frac{i-1}{i+1}\Big) \\ &= \frac{1}{a}\ln\Big(\frac{z+|z|}{z-|z|}\,i\,\frac{1+i}{i+1}\Big) \\ &= \frac{1}{a}\ln\Big(i\,\frac{z+|z|}{z-|z|}\Big) \end{aligned}$$

This is an interesting formula but we can do a little better and get some more interesting things. Set for convenience $d = d(i|z|, z)$ and $r = |z|$ and $z = x + iy$. Then

$$\begin{aligned} d &= \frac{1}{a}\ln\Big(i\,\frac{z+r}{z-r}\Big) = \frac{1}{a}\ln\Big(i\,\frac{z+r}{z-r}\,\frac{\overline{z}-r}{\overline{z}-r}\Big) \\ &= \frac{1}{a}\ln\Big(i\,\frac{z\overline{z}-r(z-\overline{z})-r^2}{z\overline{z}-r(z+\overline{z})+r^2}\Big) \\ &= \frac{1}{a}\ln\Big(i\,\frac{-2iyr}{2r^2-2rx}\Big) = \frac{1}{a}\ln\frac{y}{r-x} \\ &= \frac{1}{a}\ln\frac{y(r+x)}{r^2-x^2} = \frac{1}{a}\ln\frac{r+x}{y} \end{aligned}$$

Since the quotients are invariant under homothety we already know that d is a function of θ alone. To find this function we resort to hyperbolic functions. We have

$$\begin{aligned} e^{ad} &= \frac{y}{r-x} = \frac{r+x}{y} \\ e^{-ad} &= \frac{r-x}{y} = \frac{y}{r+x} \\ \sinh ad &= \frac{1}{2}(e^{ad}-e^{-ad}) = \frac{1}{2}\Big(\frac{r+x}{y} - \frac{r-x}{y}\Big) = \frac{x}{y} = \tan\theta \\ \cosh ad &= \frac{1}{2}(e^{ad}+e^{-ad}) = \frac{1}{2}\Big(\frac{r+x}{y} + \frac{r-x}{y}\Big) = \frac{r}{y} = \sec\theta \\ \tanh ad &= \frac{\sinh ad}{\cosh ad} = \frac{\tan\theta}{\sec\theta} = \sin\theta \end{aligned}$$

Hence

$$d = \frac{1}{a}\,\mathrm{argsinh}\ \tan\theta = \frac{1}{a}\,\mathrm{argcosh}\ \sec\theta = \frac{1}{a}\,\mathrm{argtanh}\ \sin\theta$$

We can now once again discuss equidistants. First we note that for any z_1 and z_2 we have found a Möbius transformation T which transforms the Lobachevski straight line l through z_1 and z_2 into the y-axis. An equidistant to the y-axis is a Euclidean line e_θ inclined at an angle θ from the y-axis.

The y-axis and e_θ intersect at the origin and ∞. T^{-1} will take the y-axis to (typically) a Euclidean semicircle perpendicular to the x-axis with $T^{-1}(0) = z_0$ and $T^{-1}(\infty) = z_\infty$. Then $T^{-1}[e_\theta]$ will be an arc of a Euclidean circle in the UHP from z_0 to z_∞ and inclined at an angle θ to l at z_0. Let l_1 be a Lobachevski straight line orthogonal to l meeting l and $T^{-1}[e_\theta]$ at P and Q. By conformality $T[l_1]$ is orthogonal to $T[l] = y$-axis and thus $T[l_1]$ is (typically) a Euclidean semicircle with center at the origin. But then $T[l_1]$ is also orthogonal to e_θ which means that l_1 is orthogonal to the equidistant $T[e_\theta]$ at Q. We then have

$$d(P,Q) = d(T(P), T(Q)) = \frac{1}{a}\,\text{argsinh }\tan\theta$$

which is independent of the choice of the straight line l_1 perpendicular to l, which shows $T^{-1}[e_\theta]$ is indeed an equidistant and determines the distance between $T^{-1}[e_\theta]$ and l. The reasoning reverses easily; any Euclidean circle through z_0 and z_∞ is an equidistant to l with distance given by the above formula. This proves all assertions about equidistants made up to this point

Now for horocycles. Recall that one type of horocycle was represented in the UHP by a horizontal Euclidean line. The horocyles $h_1 : \Im z = y_1$ and $h_2 : \Im z = y_2$ are conguent via the motion

$$Tz = \frac{y_2}{y_1}\,z$$

which carries h_1 to h_2. Next consider the Euclidean circle h with radius r tangent to the x-axis at x_0. Then $T_1 z = z - x_0$ is a motion carrying h to h_0 which is tangent to the x-axis at the origin. Next the motion $T_2 z = -1/z$ carries h_0 to a lircle which goes through ∞ and $i/2r$ and is perpendicular to T_2[positive y-axis] = [positive y-axis]. Thus $T_2[h_0]$ must be a horizontal Euclidean line $\Im z = 1/2r$ and thus a horocycle. Thus any Euclidean circle tangent to the x-axis is taken by a motion to a horizontal horocycle and they are all congruent. Hence these are all horocycles and all horocycles are congurent. This justifies everything we have previously said about horocycles except that they are limiting cases of circles, which we will deal with in the circle section.

Our next project is to derive the second distance formula. This requires some trickery. Recall that

$$\frac{|z_2 - z_1|^2}{(\Im z_1)(\Im z_2)}$$

is invariant under motions of the UHP, from which we conclude that

$$\frac{|z_2 - z_1|}{\sqrt{\Im z_1}\sqrt{\Im z_2}}$$

is also invariant. Let's map the straight line through (with the usual notation) z_0, z_1, z_2, z_∞ onto the y-axis with $Tz_1 = y_1 i$ and $Tz_2 = y_2 i$. ($y_1 = 1$ but we ignore this for the sake of symmetry.) Thus

$$d(z_1, z_2) = d(y_1 i, y_2 i) = \frac{1}{a}\,\ln\frac{y_2}{y_1}$$

Setting $d(z_1, z_2) = d$ we have

$$e^{ad} = \frac{y_2}{y_1} \qquad e^{-ad} = \frac{y_1}{y_2}$$
$$e^{\frac{ad}{2}} = \frac{\sqrt{y_2}}{\sqrt{y_1}} \qquad e^{\frac{-ad}{2}} = \frac{\sqrt{y_1}}{\sqrt{y_2}}$$

$$\begin{aligned} \sinh\frac{ad}{2} &= \frac{1}{2}\left(e^{\frac{ad}{2}} - e^{\frac{-ad}{2}}\right) = \frac{1}{2}\left(\frac{\sqrt{y_2}}{\sqrt{y_1}} - \frac{\sqrt{y_1}}{\sqrt{y_2}}\right) = \frac{1}{2}\left(\frac{y_2 - y_1}{(\sqrt{y_1}\sqrt{y_2}}\right) \\ &= \frac{1}{2}\frac{|y_2 - y_1|}{\sqrt{y_1}\sqrt{y_2}} \end{aligned}$$

because by construction we have $y_2 > y_1$, so

$$\begin{aligned} \sinh\frac{ad}{2} &= \frac{|iy_2 - iy_1|}{2\sqrt{y_1}\sqrt{y_2}} \\ &= \frac{|Tz_2 - Tz_1|}{2\sqrt{\Im Tz_1}\sqrt{\Im Tz_2}} \\ &= \frac{|z_2 - z_1|}{2\sqrt{\Im z_1}\sqrt{\Im z_2}} \end{aligned}$$

since the expression is invariant under motions. Hence we have the second distance formula

$$d(z_1, z_2) = \frac{2}{a}\,\text{argsinh}\,\frac{|z_2 - z_1|}{2\sqrt{\Im z_1}\sqrt{\Im z_2}}$$

In my opinion this formula is the first thing we have encountered that is not a natural outgrowth of the metric formula and the fact that certain Möbius transformations are motions of the UHP. If you didn't know this formula existed (which I ddin't) then you would have to be lucky to find it (which I was). In some circumstances it can simplify things which would be more difficult to handle with the first distance formula, for example circles.

We could, at this point, develop a lot of circle theory. However, I would prefer to wait until we have the Unit Disk (UD) model where things will be somewhat easier to see.

8. PARALLELS AT LAST

As noted in the introduction, Lobachevski Geometry was created (or discovered) in the process of studying the theory of parallels. Euclids complicated fifth postulate is equivalent to the following

P_1: Through a point outside a given line there is exactly one line parallel to the given line.

If "exactly one line" is replaced by (P_0): "no line" or (P_2): "more than one line" we obtain the three classical geometries of the two dimensional plane.

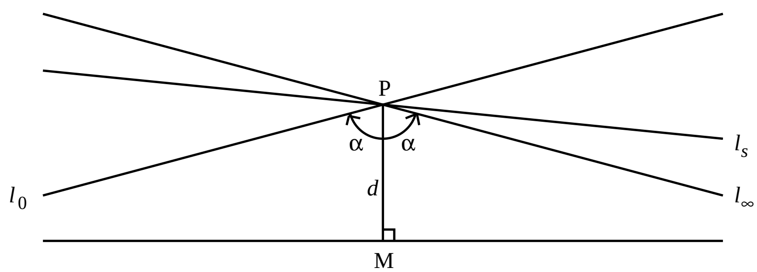

Figure 3: Straight line , point, two parallels and skew line

In the P_2 case we will obtain easier descriptions of phenomena if we refine the notion of parallel. In a P_1 geometry (Euclid) Parallel Lines are simply lines that, no matter how far extended, never meet. However, from the projective point of view we replace this by saying that two parallel lines meet at a unique point (which we imagine to be found at infinity on either end of the parallel line pair). In P_2 geometry parallel lines share a point at infinity on one side, but not on the other. In fact we can define

DEF Two lines are parallel if they meet at an infinitly distant point.

From the point of view of developing Lobachevski geometry in the plane and not within a model this might be problematical, but it is perfectly clear in the UHP model, since the infinitly distant points are the points in $\mathbb{R} \cup \{\infty\}$. Thus in the UHP model there are three cases of parallel lines

1. both lines are vertical

2. l_1 is vertical, l_2 is a semicircle perpendicular to the x-axis, and l_1 meets l_2 at either z_0 or z_∞.

3. Both l_1 and l_2 are semicircles perpendicular to the x-axis and they have a common point on the x-axis

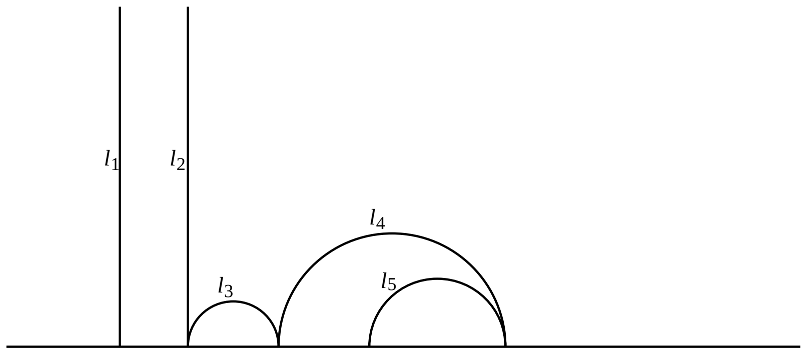

Figure 4: Parallel lines in the UHP

With this definition we see in the UHP model that through a point P outside a given line exactly two parallel lines can be drawn. In the diagrams l_0 and l_∞ are the parallels and any l_s which lies between them and thus never meets the given line is called a *skew* line.

In the illustration showing parallel lines in the UHP, l_2 is parallel to l_1 and l_3, l_3 is parallel to l_4 and l_4 is parallel to l_5; no other pairs are parallel. Note in particular that while l_4 is parallel both to l_3 and l_5, l_3 and l_5 are *not* parallel to each other.

The next illustration shows, in the UHP model, three cases of a point P outside a line l, a perpendicular dropped from P to l hitting l at M, and the parallel lines l_0 and l_∞ through P intersecting l at ∞. The *angle of parallelism* α is also shown in each case.

Since Euclid's Axioms 1-4 and our axiom P_2 are true in the UHP model, we see that Axioms 1-4 and P_2 together must be consistent. Since Euclid's Axiom 5 (and all its equivalents) are false in the UHP model, we see that Axiom 5 is actually independent of Axioms 1-4 and cannot be proved from them. Thus the UHP model settles this question completely. Lobachevski's own work did not quite settle the consistency question, since philosophers and others could argue that the contradiction might not yet have been found. Since the UHP model is a model *inside* Euclidean Geometry, a contradiction in the UHP model would force a contradiction in Euclidean Geometry. Not even a philosopher would be willing to pay that price to have Lobachevski's geometry inconsistent. Lobachevski himself did not live quite long enough to see the consistency proof. It is worth noting that the method of relative modeling for consistency and independence results starts with this material.

Now that the Philosophical Question is settled, let's ask a very interesting mathematical question. Returning to Lobachevski's picture, we can ask what the angle α is between PM and l_∞ that makes l_∞ a parallel to l. This is called

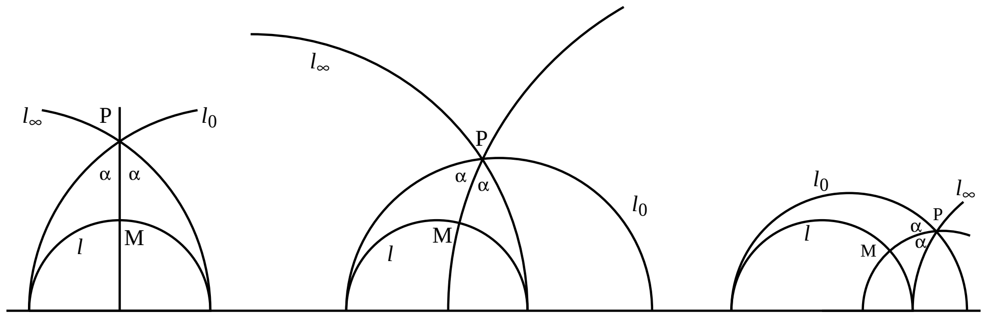

Figure 5: Lines parallel to l through P in the UHP

the *angle of parallelism.* Although it is not quite clear, one suspects that the angle of parallelism depends on the distance d from P to M; for small d, α will be close to $\pi/2$ and for large d it might decrease toward 0. Lobachevski and Janos Bolyai both discovered the formula for α, which is not so trivial, although it is not difficult if Lobachevski Trigonometry is available. I was surprised to find that the formula is an easy consequence of material from the previous section.

Next let's look at the picture in the UHP model that corresponds to the picture in the Lobachevski plane.

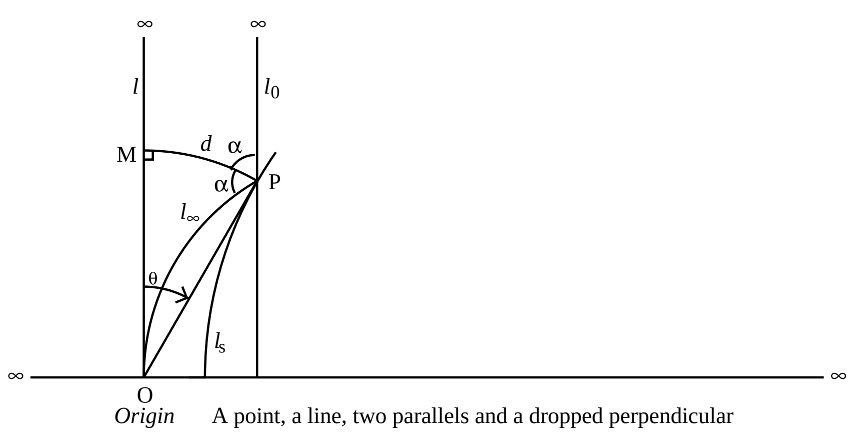

Figure 6: Straight line, point, two parallels and a skew line in UHP

Lines inclined less than α from line PM intersect l; lines inclined more than α

from PM are skew to l. Now let us use a motion to take $z_0 \to 0$, $z_\infty \to \infty$ and $M \to i$.

Next we note that we are back, as in the last section, to the situation of dropping a perpendicular from P at $z = x + iy$ to the y-axis. We recall the equations for that situation:

$$\begin{aligned} \sin\theta &= \tanh ad \\ \sec\theta &= \cosh ad \end{aligned}$$

and these will immediately solve our problem, because as the picture shows $\alpha + \theta = \pi/2$ so $\alpha = \pi/2 - \theta$.

Thus

$$\sin\alpha = \sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta = \frac{1}{\cosh ad}$$

and

$$\cos\alpha = \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta = \tanh ad$$

These equations show that

$$\begin{array}{ll} \text{as } d \to 0 \qquad & \cosh ad \to 1 \\ & \sin\alpha \to 1 \\ & \alpha \to \dfrac{\pi}{2} \end{array}$$

and

$$\begin{array}{ll} \text{as } d \to \infty \qquad & \cosh ad \to \infty \\ & \sin\alpha \to 0 \\ & \alpha \to 0 \end{array}$$

which is the behavior we predicted. Lobachevski and Bolyai's formula is a variant of those above; recall from trig

$$\begin{aligned} \tan\frac{\alpha}{2} &= \frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} = \sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} = \frac{1-\cos\alpha}{\sin\alpha} \\ &= \frac{1-\tanh ad}{\frac{1}{\cosh ad}} = \cosh ad(1-\tanh ad) \\ &= \cosh ad - \sinh ad = \frac{1}{2}[e^{ad} + e^{-ad} - (e^{ad} - e^{-ad})] \\ &= e^{-ad} \end{aligned}$$

This is the famous Lobachevski-Bolyai formula for the angle of parallelism.

We are next interested in finding a sort of interpretation of $1/a$ where a is the parameter originally introduced in the ds^2 metric formula and where $-a^2$ turned out to be the Gaussian Curvature. If we take the last formula and rewrite it we have

$$d = -\frac{1}{a}\ln\tan\frac{\alpha}{2}$$

and this will allow us to interpret $1/a$. Toward this purpose define α_0 as

$$\alpha_0 = \arctan \frac{1}{\sinh 1} \approx .705027 \text{ radians} \approx 40.395^\circ$$

Now let's compute the d which will have α_0 for its angle of parallelism. By the previous formula

$$\begin{aligned}
\tan \alpha_0 &= \frac{1}{\sinh 1} \\
\sec \alpha_0 &= \sqrt{1 + \frac{1}{\sinh^2 1}} = \frac{\cosh 1}{\sinh 1} \\
\cos \alpha_0 &= \frac{\sinh 1}{\cosh 1} \\
\sin \alpha_0 &= \frac{1}{\cosh 1} \\
\tan \frac{\alpha_0}{2} &= \frac{1 - \cos \alpha_0}{\sin \alpha_0} = \frac{1 - \frac{\sinh 1}{\cosh 1}}{\frac{1}{\cosh 1}} = \cosh 1 - \sinh 1 = e^{-1} \\
d &= -\frac{1}{a} \ln \tan \frac{\alpha_0}{2} = -\frac{1}{a}(-1) = \frac{1}{a}
\end{aligned}$$

To reiterate, set the line l_∞ at an inclination of α_0 at P and increase the distance of P from l until l_∞ is parallel to l (that is, the intersection of l_∞ and l recedes to ∞). Then the distance from P to l will be $1/a$, which can be considered a charactreristic distance for the Lobachevski plane with parameter a. In terms of the Gaussian Curvature, the characteristic distance is $1/\sqrt{-K}$.

We note that a has the dimension of inverse length and so the Gaussian Curvature $K = -a^2$ has the dimension of inverse length squared, which is obvious from other formulas for K