Vector fields

## § 1.1 Definition of a vector on a manifold M

M is a $C^\infty$ manifold， $p \in M$ ，suppose U is a neighborhood of p。

Smooth function $X_p : C^\infty(U) \to R$ satisfies

1. X(a f +b g)=a(X f)+b(X g)
2. $X_p(fg) = (X_p f)g(p) + f(p)(X_p g)$

The linear function $X_p$ is called the tangent vector on M at p。

亦即 Smooth 切向量 X 是從$C^\infty(M)$到$C^\infty(M)$的算子。

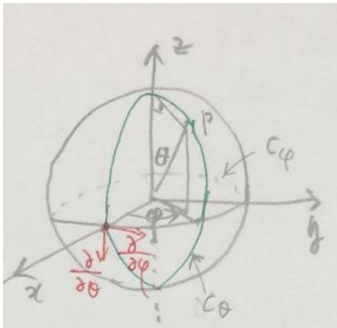
若 $f \in C^\infty(M)$則$(Xf)(p) = X_p f$ 。

Example

$X = y\dfrac{\partial}{\partial z} - z\dfrac{\partial}{\partial x}, f(x,y,z) = 2x^2y + z$ 則 $X(f) = y - z(4xy) = y - 4xyz$

過 p 點的切向量全體稱為切空間，記為$T_pM$ 。

$T_pM = \left\{\dfrac{\partial}{\partial x^1}, \dots, \dfrac{\partial}{\partial x^n}\right\}$是一個 n 維的向量空間。




例 $S^2$

$\psi : (0,\pi)\times(-\pi,\pi) \to S^2$ given by

$\psi(\theta,\varphi) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$

Parameterizes a neighorhood of the point

$(1,0,0) = \psi(\frac{\pi}{2}, 0)$

Consequently，$\left(\dfrac{\partial}{\partial\theta}\right)_{(1,0,0)} = \dot{c}_\theta(0), \left(\dfrac{\partial}{\partial\varphi}\right)_{(1,0,0)} = \dot{c}_\varphi(0)$ ，where

$c_\theta(t) = \psi(\frac{\pi}{2}+t, 0) = (\cos t, 0, -\sin t)$ ； $c_\varphi(t) = \psi(\frac{\pi}{2}, t) = (\cos t, \sin t, 0)$

Since $c_\theta$ and $c_\varphi$ are curves in $R^3$ ，$\left(\dfrac{\partial}{\partial\theta}\right)_{(1,0,0)}$ and $\left(\dfrac{\partial}{\partial\varphi}\right)_{(1,0,0)}$ can be identified with the vectors (0,0,-1) and (0,1,0)

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$ds^2 = d\theta^2 + \sin^2\theta d\varphi^2$

$x = \sin\theta\cos\varphi, y = \sin\theta\sin\varphi, z = \cos\theta$

$$\frac{\partial}{\partial\varphi} = \frac{\partial x}{\partial\varphi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial\varphi}\frac{\partial}{\partial y} + \frac{\partial z}{\partial\varphi}\frac{\partial}{\partial z}$$

$$= -\sin\theta\sin\varphi\frac{\partial}{\partial x} + \sin\theta\cos\varphi\frac{\partial}{\partial y} + 0 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$

$\frac{\partial}{\partial\varphi} = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$ is a vector field which generates rotation about the z-axis，is an isometry、and a Killing vector field that preserves the metric，i.e. $L_X g = 0$。

The flow of $\frac{\partial}{\partial\varphi} = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$ is $\varphi_t = (x\cos t - y\sin t, x\sin t + y\cos t, z)$，and the matrix of the rotation about the z-axis is $\begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\{\varphi_t \mid t \in R\} \cong SO(2,R)$

But $\frac{\partial}{\partial\theta}$ is not a Killing vector field。

§ 1.3 flow of a vector field and Lie derivative of a vector field

$\varphi : U \to M$ $\varphi_t(p) = \gamma(t), t \in (-\varepsilon, \varepsilon)$，where $\frac{d(\varphi_t(p))}{dt} = \frac{d\gamma}{dt} = X$，then $\varphi_t$ is called the flow of vector field X。

$\varphi_t$ is a 1-parameter group，$\varphi_t \circ \varphi_s(q) = \varphi_{t+s}(q)$，$\varphi_0 = identity$

Y is a $C^\infty$ vector field，the Lie derivative of Y along X is $L_X Y = \lim_{t\to 0}\frac{(\varphi_{-t})_* Y - Y}{t}$

$[X,Y] = \sum_i (XY^i - YX^i)\frac{\partial}{\partial x^i}$，then $L_X Y = [X,Y]$

Example

1. $X = -(x^2+y^2+z^2)^{-\frac{3}{2}}(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z})$，$Y = \frac{\partial}{\partial y}, r = (x^2+y^2+z^2)^{\frac{1}{2}}$

Find $L_X Y =$

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$(L_X Y)f = XYf - YXf$

$= -r^3(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z})\frac{\partial f}{\partial y}+\frac{\partial}{\partial y}(r^{-3}(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}))f$ ...(※)

$\frac{\partial}{\partial y}(r^{-3}(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}))f$

$=[(\frac{\partial}{\partial y}r^{-3})(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z})+r^{-3}\frac{\partial}{\partial y}(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z})]f$

其中$\frac{\partial}{\partial y}r^{-3}=\frac{\partial}{\partial y}(x^2+y^2+z^2)^{-\frac{3}{2}}=-\frac{3}{2}(x^2+y^2+z^2)^{-\frac{5}{2}}(2y)=-3yr^{-5}$，

$r^{-3}\frac{\partial}{\partial y}(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z})=r^{-3}[x\frac{\partial^2}{\partial x\partial y}+(\frac{\partial}{\partial y}+y\frac{\partial^2}{\partial y\partial y})+z\frac{\partial^2}{\partial y\partial z}]$ 一部分跟(※)的前項消掉，剩下$r^{-3}\frac{\partial}{\partial y}$

$(L_X Y)f = -3yr^{-5}(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z})+r^{-3}\frac{\partial}{\partial y}f$

$= r^{-5}\{-3xy\frac{\partial}{\partial x}+(r^2-3y^2)\frac{\partial}{\partial y}-3yz\frac{\partial}{\partial z}\}f$

So $L_X Y = r^{-5}\{-3xy\frac{\partial}{\partial x}+(r^2-3y^2)\frac{\partial}{\partial y}-3yz\frac{\partial}{\partial z}\}$

2. $X = y\frac{\partial}{\partial z}-z\frac{\partial}{\partial y}$，$Y = z\frac{\partial}{\partial x}-x\frac{\partial}{\partial z}$

(1) 求 X 的 flow $\varphi_t$，Y 的 flow $\psi_t$

(2) 求[X,Y]=

(3) 驗證$L_X Y = \lim_{t\to 0}\frac{(\varphi_{-t})_* Y - Y}{t} = [X,Y]$

(1) $\frac{d\varphi_t(p)}{dt} = X_{\varphi_t(p)}, \varphi_{0(p)} = p$

$$\begin{cases} \dot{\varphi}_t^1(p) = X^1(\varphi_{t(p)}) = 0 \\ \dot{\varphi}_t^2(p) = X^2(\varphi_{t(p)}) = -\varphi_t^3 \quad \Rightarrow \varphi_t^1 = c, \ddot{\varphi}_t^2 = -\dot{\varphi}_t^3 = -\varphi_t^2 \\ \dot{\varphi}_t^3(p) = X^3(\varphi_{t(p)}) = \varphi_{t(p)}^2 \end{cases}$$

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Then $\varphi_t^1 = C,\ \varphi_t^2 = A\cos t + B\sin t, \varphi_t^3 = A\sin t - B\cos t$

A，B，C are function of p=(x y z)

And $\varphi_0(x,y,z) = (x,y,z)$ ，so C=x，A=y，B=-z

$\varphi_t(x,y,z) = (x, y\cos t - z\sin t, y\sin t + z\cos t)$

$\varphi_{-t}(x,y,z) = (x, y\cos t + z\sin t, -y\sin t + z\cos t)$ = $(\hat{x}, \hat{y}, \hat{z})$ ...(*)

同理

$\psi_t(x,y,z) = (x\cos t + z\sin t, y, -x\sin t + z\cos t)$

(2) $[X,Y] = (XY^i - YX^i)\frac{\partial}{\partial^i} = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$

(3) $(\varphi_{-t})_* Y = Y$ (in coordinate $\hat{x}, \hat{y}, \hat{z}$ )

由(*)解出 $x = \hat{x}, y = \hat{y}\cos t - \hat{z}\sin t, z = \hat{y}\sin t + \hat{z}\cos t$

$$Y = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} = (\hat{y}\sin t + \hat{z}\cos t)\frac{\partial}{\partial \hat{x}} - \hat{x}(\sin t\frac{\partial}{\partial \hat{y}} + \cos t\frac{\partial}{\partial \hat{z}})\Big|_{at(\hat{x},\hat{y},\hat{z})}$$

Then $L_X Y = \frac{d}{dt}((\varphi_{-t})_* Y)_{t=0}$

$$= (\hat{y}\cos t - \hat{z}\sin t)\frac{\partial}{\partial \hat{x}} - (\hat{x}\cos t)\frac{\partial}{\partial \hat{y}} + (\hat{x}\sin t)\frac{\partial}{\partial \hat{z}}\Big|_{t=0} = \hat{y}\frac{\partial}{\partial \hat{x}} - \hat{x}\frac{\partial}{\partial \hat{y}}$$

Exercises

(1) $X = -(x^2+y^2+z^2)^{-\frac{3}{2}}(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z})$ ，$Y = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$ ，find $L_X Y =$ 0

(2) $X = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + x(y+1)\frac{\partial}{\partial z}$ ，$Y = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}$ 求

(a) [X, Y ]=

(b) 求 X 的 flow $\varphi_t$ 並驗證 $\frac{d}{dt}((\varphi_{-t})_* Y)\Big|_{t=0} = [X,Y]$

$[X,Y] = (XY^i - YX^i)\partial_i$

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$$=(XY^1-YX^1)\frac{\partial}{\partial x}+(XY^2-YX^2)\frac{\partial}{\partial y}+(XY^3-YX^3)\frac{\partial}{\partial z}$$

$( X^i=(1,1,x(y+1)), Y^i=(1,0,y))$

$XY^1=X(1)=0, YX^1=Y(1)=0$

$XY^2=X(0)=0, YX^2=Y(1)=0$

$XY^3=X(y)=1, YX^3=Y(x(y+1))=y+1$

所以[X,Y]=$-y\frac{\partial}{\partial z}$

以下求 X 的 flow $\varphi_t$ ，並驗證$\frac{d}{dt}((\varphi_{-t})_*Y)\big|_{t=0}=[X,Y]$

$\dot{\varphi}_t^1=1, \dot{\varphi}_t^2=1$， $\varphi_0(x,y,z)=(x,y,z)$ 所以$\varphi_t^1=t+x, \varphi_t^2=t+y$

$\dot{\varphi}_t^3=(t+x)(t+y+1), \varphi_0^3=z$ ，所以$\varphi_t^3=\frac{1}{3}t^3+\frac{1}{2}(x+y+1)t^2+x(y+1)t+z$

$\varphi_{-t}(x,y,z)=(-t+x,-t+y,-\frac{1}{3}t^3+\frac{1}{2}(x+y+1)t^2-x(y+1)t+z)$=$(\bar{x},\bar{y},\bar{z})$

$(\varphi_{-t})_*Y=Y$ (in coordinate of $\bar{x},\bar{y},\bar{z}$ )，

由 chain rule

$\partial_x=\frac{\partial\bar{x}}{\partial x}\partial_{\bar{x}}+\frac{\partial\bar{y}}{\partial x}\partial_{\bar{y}}+\frac{\partial\bar{z}}{\partial x}\partial_{\bar{z}}=\partial_{\bar{x}}$+$[\frac{1}{2}t^2-(y+1)t]\partial_{\bar{z}}$ ， $y+1=\bar{y}+t+1$

$\partial_z=\frac{\partial\bar{x}}{\partial z}\partial_{\bar{x}}+\frac{\partial\bar{y}}{\partial z}\partial_{\bar{y}}+\frac{\partial\bar{z}}{\partial z}\partial_{\bar{z}}=\partial_{\bar{z}}$

所以$(\varphi_{-t})_*Y=\partial_{\bar{x}}+[\frac{1}{2}t^2-(\bar{y}+t+1)t]\partial_{\bar{z}}+(\bar{y}+t)\partial_{\bar{z}}$

$\frac{d}{dt}((\varphi_{-t})_*Y)_{t=0}=-(\bar{y}+1)\partial_{\bar{z}}+\partial_{\bar{z}}=-\bar{y}\partial_{\bar{z}}=[X,Y]_{at(\bar{x},\bar{y},\bar{z})}$

$\varphi_t(x,y,z)=(t+x,t+y,\frac{1}{3}t^3+\frac{1}{2}t^2(x+y+1)+tx(y+1)+z)$

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3. $X_1 = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, X_2 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}, X_3 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$

(a) Compute $[X_i, X_j]$

(b) Show that span $\{X_1, X_2, X_3\}$ is a Lie subalgebra of $\chi(R^3)$，isomorphic to

$(R^3, X)$

(c) Compute flows of $X_i$

(d) $\varphi_{i,\frac{\pi}{2}} \circ \varphi_{j,\frac{\pi}{2}} \neq \varphi_{j,\frac{\pi}{2}} \circ \varphi_{i,\frac{\pi}{2}}$ for $i \neq j$

(e) Calculate $\frac{\partial}{\partial \varphi}(r)$ where $r = \sqrt{x^2 + y^2 + z^2}$

解

(a) $[X_1, X_2] = -X_3, [X_1, X_3] = X_2, [X_2, X_3] = -X_1$

(b) 假設 $V := span\{X_1, X_2, X_3\}$ with [，]

$F : V \to R^3$

$F(aX_1 + bX_2 + cX_3) = (a, -b, c)$ is a bijective

Show that it is a Lie algebra homeomorphism

即 $F([X_i, X_j]) = ... = F(X_i) \times F(X_j)$

例如 $F([X_1, X_2]) = F(-X_3) = (0,0,-1)$

$F(X_1) \times F(X_2) = (1,0,0) \times (0,-1,0) = (0,0,-1)$

So F is a homomorphism，$\therefore \mathrm{V} \cong (\mathrm{R}^3, \mathrm{X})$

(c) $\varphi_{1,t}(x, y, z) = (x, y\cos t - z\sin t, y\sin t + z\cos t)$

$\varphi_{2,t}(x, y, z) = (x\cos t + z\sin t, y, -x\sin t + z\cos t)$

$\varphi_{3,t}(x, y, z) = (x\cos t - y\sin t, x\sin t + y\cos t, z)$

(d) 略

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(e) $M_x(r)=-z\dfrac{\partial r}{\partial y}+y\dfrac{\partial r}{\partial z}=-z\times\dfrac{1}{2}(2y)(x^2+y^2+z^2)^{-\frac{1}{2}}+y\times\dfrac{1}{2}(2z)(x^2+y^2+z^2)^{-\frac{1}{2}}=0$

§ complete vector fields

$\varphi_t : W \to M$

(1) Local diffeomorphism

(2) $(\varphi_t \circ \varphi_s)(q)=\varphi_{t+s}(q)$---(*)

$\{\varphi_t : I \to M\}_{t\in I}$ ， $I=(-\varepsilon,\varepsilon)$ 滿足(*) 稱為 local one-parameter group (of diffeomorphism)

例

1. $\phi$:$\mathrm{R}\times\mathrm{R}^2 \to R^2$ ，(t,(x,y))→(x+t,y-3t)

是否定義一個 one-parameter group？

$\phi(t,u)=\varphi_t(u)$ 則

$\varphi_0(x,y)=(x,y)\Rightarrow\varphi_0=id$

$\varphi_t(x,y)=(x+t,y-3t)$

驗證 $\varphi_s\circ\varphi_t(x,y)=\ldots=\varphi_{s+t}(x,y)$ ，

$(\varphi_s^{-1}\circ\varphi_s)(x,y)=\varphi_s^{-1}(x+s,y-3s)=(x,y)$ ，$\therefore \varphi_s^{-1}(x,y)=(x-s,y+3s)=\varphi_{-s}(x,y)$

所以 $\phi$ 定義一個 one-parameter group

2. $\phi: R\times R^2 \to R^2$

$\phi(t,(x,y))=(tx,y-x)$ 沒定義一個 one-parameter group

因為 $\varphi_t(x,y)=(tx,y-x)$ 則 $\varphi_0(x,y)=(0,y-x)\neq(x,y)$

3. $\phi$:$\mathrm{R}\times\mathrm{R}^2 \to R^2$

$\phi(t,(x,y))=(x\cos 2t-y\sin 2t, x\sin 2t+y\cos 2t+t)$ 沒定義一個 one-parameter group

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$\varphi_t(x,y)=(x\cos 2t-y\sin 2t, x\sin 2t+y\cos 2t+t)$

$\varphi_0(x,y)=(x,y)$

但是$\varphi_{s+t}\neq\varphi_s\circ\varphi_t$(要驗證)

一個向量場 X，其 local flow 定義一個 one-parameter group of diffeomorphism，

$(\varphi_t\circ\varphi_s)(q)=\varphi_{t+s}(q)$ ，$\varphi_0=identity$

(all solution curves exist for all time。換句話說 flow 定義到整個 R)

則稱 X 為完備的(complete)向量場。

一個 one-parameter group 的軌跡(orbit)是向量場的積分曲線(a curve tangent to vector field)

Example

1. $X\in\chi(R)$ ，$X=x^2\dfrac{d}{dx}$ is incomplete

2. $X=x\dfrac{\partial}{\partial x}$ on M=R is complete

3. M=R[2] $X_1=\dfrac{\partial}{\partial x}$ ，$X_2=x\dfrac{\partial}{\partial x}+y\dfrac{\partial}{\partial y}$ ，$X_3=y\dfrac{\partial}{\partial x}-x\dfrac{\partial}{\partial y}$ are all complete

解

1. $X=x^2\dfrac{d}{dx}$ ，求$\dot{x}=x^2$的積分曲線

$\dfrac{dx}{dt}=x^2$ ，$\dfrac{dx}{x^2}=dt$ 兩邊積分$-\dfrac{1}{x}=t+c$ ，$x=\dfrac{-1}{t+c}$ $\therefore\varphi_t=\dfrac{-1}{t+c}$ ，$\varphi_0(x)=x$

$-\dfrac{1}{c}=x$ $\therefore\varphi_t=\dfrac{x}{1-tx}:W\to M$

設 W=(a,b)，a>0

$F:W\times I\to M$ F(q,0)=q，$\dfrac{\partial F}{\partial t}(q,t)=X_{F(q,t)}$則 local flow 只能延伸到$W\times(-\infty,\dfrac{1}{b})$

另一種說法：

For initial condition $x(0)=x_0\neq 0$ ，$x(t)=\dfrac{x_0}{1-tx_0}$在$t=\dfrac{1}{x_0}$沒有定義。

所以 X 非完備向量場。

力學中 大部分的(Hamiltonian)向量場不是完備的向量場。

Vector fields

4. $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ on $R^2$

求 X 的 flow,積分曲線 並且說明 X 是完備向量場

$\varphi_t = (\varphi_1(t), \varphi_2(t))$

$\dot{\varphi}_1(t) = \varphi_1(t)$，$\varphi_1(t) = ae^t$

$\dot{\varphi}_2(t) = \varphi_2(t)$，$\varphi_2(t) = be^t$

$\because \varphi_0(x, y) = (x, y)$，所以 a=x，b=y

$\varphi_t(x, y)=(xe^t, ye^t)$，$\varphi_{s+t} = \varphi_s \circ \varphi_t$ 是 1-parameter group with $\varphi_0 = identity$

Theorem

$X, Y \in \chi(M)$ are two complete vector fields with flows $\varphi, \psi$ then $\varphi, \psi$ commute $\Leftrightarrow$ [X Y]=0

Proposition

Every vector field on a compact manifold is complete。

Exercises

[D01] p.33 Ans at p.333

Let $X, Y \in \mathfrak{X}(M)$ be two complete vector fields with flows $\psi, \phi$. Show that:

(a) given a diffeomorphism $f : M \to M$, we have $f_*X = X$ if and only if $f \circ \psi_t = \psi_t \circ f$ for all $t \in \mathbb{R}$;

(b) $\psi_t \circ \phi_s = \phi_s \circ \psi_t$ for all $s, t \in \mathbb{R}$ if and only if $[X, Y] = 0$.

(b)

$\Rightarrow$ $\psi_t \circ \phi_s = \phi_s \circ \psi_t$ for $\forall s, t \in R$

By (a) $(\psi_t)_* Y = Y$，$[X, Y] = L_X Y = \frac{d}{dt}((\psi_{-t})_* Y)\big|_{t=0} = \frac{d}{dt} Y\big|_{t=0} = 0$

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If, on the other hand, $[X, Y] = 0$ then

$$\frac{d}{dt}((\psi_t)_* Y) = \frac{d}{d\varepsilon}((\psi_{t+\varepsilon})_* Y)_{|\varepsilon=0} = \frac{d}{d\varepsilon}((\psi_t)_*(\psi_\varepsilon)_* Y)_{|\varepsilon=0}$$
$$= (\psi_t)_* \frac{d}{d\varepsilon}((\psi_\varepsilon)_* Y)_{|\varepsilon=0} = -(\psi_t)_* L_X Y = 0.$$

Since $(\psi_0)_* Y = Y$, we conclude that $(\psi_t)_* Y = Y$ for all $t \in \mathbb{R}$. Therefore $\psi_t \circ \phi_s = \phi_s \circ \psi_t$ for all $s, t \in \mathbb{R}$.

§ V.I. Arnold

$\frac{d\varphi}{dt} = V(t, \varphi(t))$ is the phase velocity vector field，then the phase flow is $\varphi_t(x)$

with $\varphi_0(q) = q$

Example

1. $\dot{x} = kx$

$\dot{\varphi} = k\varphi$ with $\varphi_0 = identity$，then $\varphi_t(x) = xe^{kt}$，the phase flow is the group

$\{xe^{kt}\}$，which is a one-parameter diffeomorphism group。

參考[https://jmath2020.neocities.org/DifferentialEquation/DEflows.pdf]
關於一個微分方程的 flow

一個 vector field 與一個微分方程應該是指同一現象。

2. Show that the vector field $X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$ is complete on $R^2$。

$\begin{cases} \dot{x} = -y \\ \dot{y} = x \end{cases}$ with $x(0) = x_0, y(0) = y_0$，then $\ddot{y} = \dot{x} = -y$

$y(t) = x_0 \sin t + y_0 \cos t, x(t) = x_0 \cos t - y_0 \sin t$

$(x(t))^2 + (y(t))^2 = x_0^2 + y_0^2$ the integral curves are concentric circles。

The one-parameter group：

$\varphi_t(x, y) = (x\cos t - y\sin t, x\sin t + y\cos t), \varphi_0(x, y) = (x, y)$

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$\{\varphi_t | t \in \mathrm{R}\}$=SO(2,R)

§ $X = xz\frac{\partial}{\partial x} + yz\frac{\partial}{\partial y} - (x^2+y^2)\frac{\partial}{\partial z}$ ，求通過 A(2,-1,1)的 integral curve

$\frac{dx}{xz} = \frac{dy}{yz} = \frac{dz}{-(x^2+y^2)}$

由 $\frac{dx}{x} = \frac{dy}{y} \Rightarrow y = c_1 x$

$\frac{xdx}{x^2} = \frac{ydy}{y^2} = \frac{dz}{-(x^2+y^2)}$ ， $\frac{d(x^2+y^2)}{2(x^2+y^2)} = \frac{dz}{-(x^2+y^2)} \Rightarrow x^2+y^2+z^2 = c_2$

A 點代入 得

$$C:\begin{cases} y = -\frac{1}{2}x \\ x^2+y^2+z^2 = 6 \end{cases}$$

$P\frac{\partial z}{\partial x} + Q\frac{\partial z}{\partial y} = R$ is a Lagrange linear equation，then $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

§ $\dot{X} = AX$

$\exp : \eta \to G$ with $V \to \varphi_1(e)$

其中 $\varphi_t$ 是 left-invariant vector field $X^V$ 的 flow，then $\varphi_t(e) = \exp(tV)$

Consider $h(t) = e^{tA}$ ，then h(0)=I

$\frac{d}{dt}h(t) = Ae^{tA} = Ah(t)$ ，so h(t) is the flow of $X^A$ at e (i.e. $h(t) = \varphi_t(e)$ )

Then $\exp A = \varphi_1(e) = e^A$

$\exp A = \sum_{k=0}^{\infty}\frac{1}{k!}A^k$ ， $M_{n\times n}(R) \xrightarrow{\exp} GL(n,R)$

$\varphi_A(t) = \exp(tA)$ is a one-parameter group with $\frac{d}{dt}(\exp tA)|_{t=0} = A$

So $M_{n\times n}(R)$ is the Lie algebra of GL(n ,R) with [ ]。

Vector fields

Example

$$X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$$

(1) 求 X 的 flow

(2) 求 X 的積分線

(3) 說明 X 是完備向量場

$\varphi_t$ is the flow of $X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$ with $\frac{d\varphi_t(q)}{dt} = X_{\varphi_t(q)}$ and $\varphi_0(q) = q$

$$\begin{cases} \dot{x} = -y \\ \dot{y} = x \end{cases}$$ with $x(0) = x_0, y(0) = y_0$

$\ddot{y} = \dot{x} = -y$ then $y(t) = x_0 \sin t + y_0 \cos t$ ， $x(t) = x_0 \cos t - y_0 \sin t$

$$\varphi_t(x, y) = (x_0 \cos t - y_0 \sin t, x_0 \sin t + y_0 \cos t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

The vector field is complete。

The integral curves are concentric circles $x^2 + y^2 = {x_0}^2 + {y_0}^2$ 。

$\{\varphi_t | t \in R\} = SO(2, R)$ 是一個旋轉群