

[Spacetime and Geometry] Sean Carroll

Ch 3 p.107

求 geodesic equation 有多種方法，用 Euler-Lagrange 方程式是其一。

c.f. Geodesics

這裡考慮對 proper time functional $\tau = \int \left(-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \right)^{\frac{1}{2}} d\lambda$ 作變分。

$$I = \frac{1}{2} \int g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} d\tau \quad (\text{這裡把參數從一般的 } \lambda \text{ 改成 } \tau, \text{ 不十分明白。})$$

考慮路徑的無窮小變分(infinitesimal variations)下積分的變化

$$x^\mu \rightarrow x^\mu + \delta x^\mu$$

$$g_{\mu\nu} \rightarrow g_{\mu\nu} + (\partial_\sigma g_{\mu\nu}) \delta x^\sigma \quad (\text{Taylor expansion in curved spacetime})$$

$$\delta I = \frac{1}{2} \int \left[\partial_\sigma g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \delta x^\sigma + g_{\mu\nu} \frac{d(\delta x^\mu)}{d\tau} \frac{dx^\nu}{d\tau} + g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{d(\delta x^\nu)}{d\tau} \right] d\tau$$

後面兩項作分部積分，例如

$$\begin{aligned} \frac{1}{2} \int g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{d(\delta x^\nu)}{d\tau} d\tau &= \frac{1}{2} \left\{ g_{\mu\nu} \frac{dx^\mu}{d\tau} \delta x^\nu \Big|_a^b - \int \delta x^\nu \frac{d}{d\tau} \left(g_{\mu\nu} \frac{dx^\mu}{d\tau} \right) d\tau \right. \\ &= -\frac{1}{2} \int \left[g_{\mu\nu} \frac{d^2 x^\mu}{d\tau^2} + \frac{dg_{\mu\nu}}{d\tau} \frac{dx^\mu}{d\tau} \right] \delta x^\nu d\tau \end{aligned}$$

Where the variation δx^μ vanish at the end points of the path。

$$= -\frac{1}{2} \int \left[g_{\mu\nu} \frac{d^2 x^\mu}{d\tau^2} + \partial_\sigma g_{\mu\nu} \frac{dx^\sigma}{d\tau} \frac{dx^\mu}{d\tau} \right] \delta x^\nu d\tau$$

After rearranging some dummy indices，

$$\delta I = - \int \left[g_{\mu\sigma} \frac{d^2 x^\mu}{d\tau^2} + \frac{1}{2} (\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}) \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \right] \delta x^\sigma d\tau.$$

Since we are searching for stationary points，we want δI to vanish for any variation δx^σ ，this implies

$$g_{\mu\sigma} \frac{d^2 x^\mu}{d\tau^2} + \frac{1}{2} (\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}) \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0,$$

and multiplying by the inverse metric $g^{\rho\sigma}$ finally leads to

$$\frac{d^2 x^\rho}{d\tau^2} + \frac{1}{2} g^{\rho\sigma} (\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}) \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0.$$

The geodesic equation。

§ 弧長的變分

$\gamma: [0, l] \rightarrow M$ 是 M 上的可微曲線。

$\hat{\gamma}: [0, l] \times (-\varepsilon, \varepsilon) \rightarrow M$ ，使得 $\hat{\gamma}(t, 0) = \gamma(t)$ ，稱 $\hat{\gamma}$ 為 γ 的變分 (variation)

若 $\hat{\gamma}(0, s) = \gamma(0), \hat{\gamma}(l, s) = \gamma(l)$, for $\forall s \in (-\varepsilon, \varepsilon)$ 則稱 $\hat{\gamma}$ 為 proper variation

$$\gamma_s = \gamma_s(t) \equiv \hat{\gamma}(t, s)$$

$$L(s) = \int_0^l \left| \frac{d\gamma_s}{dt} \right| dt = \int_0^l \langle T, T \rangle^{\frac{1}{2}} dt \text{ 是 } \gamma_s \text{ 的長度。}$$

$$\text{Lemma 1} \quad L'(s) = \int_0^l \frac{\partial}{\partial s} \left| \frac{d\gamma_s}{dt} \right| dt$$

證明

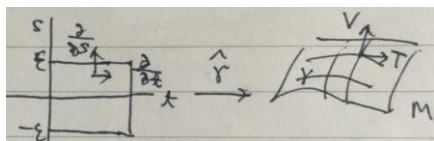
$$\begin{aligned} L'(s) &= \lim_{h \rightarrow 0} \frac{L(s+h) - L(s)}{h} = \lim_{h \rightarrow 0} \frac{\int_0^l \left(\left| \frac{d\gamma_{s+h}}{dt} \right| - \left| \frac{d\gamma_s}{dt} \right| \right) dt}{h} \\ &= \int_0^l \lim_{h \rightarrow 0} \frac{\left| \frac{d\gamma_{s+h}}{dt} \right| - \left| \frac{d\gamma_s}{dt} \right|}{h} dt \end{aligned}$$

(在 $[0, l]$ 有 finite length 且 $\frac{\partial}{\partial s} \left| \frac{d\gamma_s}{dt} \right|$ 存在 則 $\int_0^l, \lim_{h \rightarrow 0}$ 可交換。)

$$= \int_0^l \frac{\partial}{\partial s} \left| \frac{d\gamma_s}{dt} \right| dt$$

Lemma 2 $V(t), W(t)$ 是沿 γ 的可微向量場，則

$$\frac{d}{dt} \langle V, W \rangle = \left\langle \frac{DV}{dt}, W \right\rangle + \left\langle V, \frac{DW}{dt} \right\rangle$$



取 $T = \frac{\partial \hat{\gamma}}{\partial t} = \hat{\gamma}_* \left(\frac{\partial}{\partial t} \right), V = \frac{\partial \hat{\gamma}}{\partial s} = \hat{\gamma}_* \left(\frac{\partial}{\partial s} \right)$ 是定義在 $[0, l] \times (-\varepsilon, \varepsilon)$ 上的兩向量場。

$$\text{令 } L(s) = \int_0^l \left| \frac{d\gamma_s}{dt} \right| dt = \int_0^l \langle T, T \rangle^{\frac{1}{2}} dt$$

假設在 γ 上 $(s=0) |T| = 1$ ，則

$$L'(s) = \int_0^l V \langle T, T \rangle^{\frac{1}{2}} dt = \int_0^l \frac{2 \langle \nabla_V T, T \rangle}{2 \langle T, T \rangle^{\frac{1}{2}}} dt \quad (\text{由 lemma 1})$$

$$\left(\int_0^l V \langle T, T \rangle^{\frac{1}{2}} dt = \int_0^l \frac{\partial}{\partial s} \langle T, T \rangle^{\frac{1}{2}} dt, \quad \frac{\partial}{\partial s} \langle T, T \rangle^{\frac{1}{2}} = \frac{2 \langle \frac{\partial}{\partial s} T, T \rangle}{2 \langle T, T \rangle^{\frac{1}{2}}} \right)$$

$$L'(0) = \int_0^l \langle \nabla_V T, T \rangle dt$$

(M 上的 torsion 張量 $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$)

此處 ∇ is torsion-free (無撓) 即 $T=0$, 且 $[T,V] = \hat{\gamma}_* \left[\frac{\partial}{\partial t}, \frac{\partial}{\partial s} \right] = 0$

$$T \langle V, T \rangle = \langle \nabla_T V, T \rangle + \langle V, \nabla_T T \rangle$$

$$\int_0^l \langle \nabla_V T, T \rangle dt = \int_0^l \langle \nabla_T V, T \rangle dt = \int_0^l T \langle V, T \rangle - \langle V, \nabla_T T \rangle dt$$

$$= \langle V, T \rangle \Big|_0^l - \int_0^l \langle V, \nabla_T T \rangle dt$$

此式依賴 V , 故寫成

$$L'_V(0) = \langle V, T \rangle \Big|_0^l - \int_0^l \langle V, \nabla_T T \rangle dt \quad \text{是弧長的一階變分式。}$$

Do Carmo 書的寫法是

$$L'(0) = - \int_0^l \langle A(s), V(s) \rangle, \quad \text{其中 } A(s) = \frac{D}{ds} \frac{\partial \gamma}{\partial s}(s,0) \text{ 稱為曲線 } \gamma \text{ 的加速向量。}$$

定理一.

$$(M,g) \text{ 上的一條曲線 } \gamma = \gamma(t), \quad \text{設 } |T| = \left| \frac{d\gamma}{dt} \right| = 1$$

$\hat{\gamma}$ 是 γ 的變分, V 為變分向量場 且 $V(0,0) = V(l,0) = 0$

則 $L'_V(0) = 0 \Leftrightarrow \gamma$ 是測地線 (即在 γ 上 $\nabla_T T = 0$)

給定一條測地線, 上面的 Jacobi 場就是這條測地線上的測地變分場。

幾何上來說, 它刻畫了流形上一條測地線「變動」到附近測地線的趨勢。

§ 弧長的第二變分式

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z$$

$$\text{截曲率(sectional curvature)} \quad K(X, Y) = \frac{\langle R(Y, X)X, Y \rangle}{|X \wedge Y|^2}$$

$$\text{Lemma 3} \quad \frac{D}{ds} \frac{\partial \gamma}{\partial t} = \frac{D}{\partial t} \frac{\partial \gamma}{\partial s}$$

$$\text{以下 } T = \frac{\partial}{\partial t}, V = \frac{\partial}{\partial s}$$

$$L'(s) = \int_0^l V \langle T, T \rangle^{\frac{1}{2}} dt = \int_0^l \frac{\langle \nabla_V T, T \rangle}{\langle T, T \rangle^{\frac{1}{2}}} dt$$

$$L''(s) = \left(\frac{d^2}{ds^2} L \right)_{s=0} = \int_0^l V \left(\frac{\langle \nabla_V T, T \rangle}{\langle T, T \rangle^{\frac{1}{2}}} \right) dt = \int_0^l \left(\frac{V \langle \nabla_V T, T \rangle}{\langle T, T \rangle^{\frac{1}{2}}} - \frac{\langle \nabla_V T, T \rangle^2}{\langle T, T \rangle^{\frac{3}{2}}} \right) dt$$

當 $s=0$ 時， $\langle T, T \rangle = 1, \therefore \nabla_T T = 0$

$$T \langle \nabla_V V, T \rangle = \langle \nabla_T \nabla_V V, T \rangle + \langle \nabla_V V, \nabla_T T \rangle = \langle \nabla_T \nabla_V V, T \rangle \dots (*)$$

$$L''(0) = \left(\frac{d^2}{ds^2} L \right)_{s=0} = \int_0^l V \left(\frac{\langle \nabla_V T, T \rangle}{\langle T, T \rangle^{\frac{1}{2}}} \right) dt \Big|_{s=0} = \int_0^l (V \langle \nabla_V T, T \rangle - \langle \nabla_V T, T \rangle^2) dt$$

In general $[T, V]=0$ ，所以 $\nabla_V T = \nabla_T V$

$$\text{其中 } V \langle \nabla_V T, T \rangle = \langle \nabla_V \nabla_V T, T \rangle + \langle \nabla_V T, \nabla_V T \rangle = \langle \nabla_V \nabla_T V, T \rangle + |\nabla_T V|^2$$

因為 $[V, T]=0$

$$R(V, T)V = \nabla_V \nabla_T V - \nabla_T \nabla_V V - \nabla_{[V, T]} V = \nabla_V \nabla_T V - \nabla_T \nabla_V V \text{ 移項}$$

$$\nabla_V \nabla_T V = R(V, T)V + \nabla_T \nabla_V V$$

兩邊對 T 做內積，且由(*)

$$\text{所以 } \langle \nabla_V \nabla_T V, T \rangle = \langle R(V, T)V, T \rangle + \langle \nabla_T \nabla_V V, T \rangle = \langle R(V, T)V, T \rangle + T \langle \nabla_V V, T \rangle$$

$$\text{所以 } V \langle \nabla_V T, T \rangle = \langle R(V, T)V, T \rangle + T \langle \nabla_V V, T \rangle + |\nabla_T V|^2$$

$$\text{又 } T \langle V, T \rangle = \langle \nabla_T V, T \rangle + \langle V, \nabla_T T \rangle = \langle \nabla_T V, T \rangle = \langle \nabla_V T, T \rangle$$

$$\begin{aligned}
L''(0) &= \int_0^l (\langle R(V,T)V, T \rangle + T \langle \nabla_V V, T \rangle + |\nabla_T V|^2 - (T \langle V, T \rangle)^2) dt \\
&= \int_0^l (|\nabla_T V|^2 - \langle R(V,T)T, V \rangle) dt - \int_0^l (T \langle V, T \rangle)^2 dt + (\langle \nabla_V V, T \rangle) \Big|_0^l
\end{aligned}$$

定理

γ 是 (M, g) 上的一條測地線，設 $|T|=1$

對 γ 做變分 $\hat{\gamma}$ ， V 是變分向量場，設 $\langle V, T \rangle$ 沿 γ 為常數 且在 γ 兩端點上 $V=0$

$$\text{則 } L''(0) = \int_0^l (|\nabla_T V|^2 - \langle R(V,T)T, V \rangle) dt = - \int_0^l \langle \nabla_T^2 V + R(V,T)T, V \rangle dt$$

這裡 $K(T, V) \equiv \langle R(V,T)T, V \rangle$ 是截曲率。

c.f. JacobFields(測地線的變分)

設 $p, q \in M$ ， $\Omega_{pq} := \{\gamma: \gamma \text{ 為 } M \text{ 上連接 } p \text{ 到 } q \text{ 的 } C^\infty \text{-曲線}\}$

$$\text{長度 } L(s) = \int_0^l \left| \frac{d\gamma_s}{dt} \right| dt, \text{ 能量 } E(s) = \int_0^l \left| \frac{d\gamma_s}{dt} \right|^2 dt.$$

在 Ω_{pq} 中

1. γ 是 E 的 stationary point $\Leftrightarrow \gamma$ 是測地線 ($\Rightarrow \gamma$ 是 L 的 stationary point)。
2. 設 γ 是測地線，在 Ω_{pq} 中考慮某二階變分，則 $L''(0) \leq 0 \Leftrightarrow E''(0) \leq 0$

p.194~196 有關於測地線穩定的說明。

§ Synge 定理

封閉的正曲率流形若為偶數維且可定向 則必為單連通。