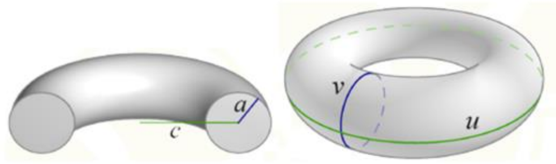
§ Torus



[the torus]

§ equation

The parameter equations for a torus are：

$$\begin{cases} x = (R + r\cos\theta)\cos\phi \\ y = (R + r\cos\theta)\sin\phi \\ z = r\sin\theta \end{cases}$$ where $\theta \in [0, 2\pi)$ ：The angle around the tube (poloidal angle)

$\phi \in [0, 2\pi)$ ：The angle around the central axis of the torus(toroidal angle)

§ curvatures

The Gaussian curvature $K = \dfrac{\cos\theta}{r(R + r\cos\theta)}$

1. **Positive Curvature**: On the outer part of the torus (where $\cos\theta > 0$), the Gaussian curvature is positive.
2. **Negative Curvature**: On the inner part of the torus (where $\cos\theta < 0$), the Gaussian curvature is negative.
3. **Zero Curvature**: At the top and bottom of the torus (where $\cos\theta = 0$), the Gaussian curvature is zero.

Total curvature $\int_{torus} KdA = 2\pi\chi$

For a torus，the Euler characteristic χ is zero， so the total curvature is zero。This reflects the fact that the positive curvature on the outer part of the torus cancels out the negative curvature on the inner part。

The mean curvature H at a point is given by $H = \dfrac{R + 2r\cos\theta}{2r(R + r\cos\theta)}$

- On the **outer part** of the torus ($\cos\theta > 0$), the mean curvature H is **positive**.
- On the **inner part** of the torus ($\cos\theta < 0$), the mean curvature H can be **negative** or **positive**, depending on the values of R and r.

Special case：

1. At the top and nottom $\theta = \dfrac{\pi}{2}$ or $\dfrac{3\pi}{2}$ $H = \dfrac{1}{2r}$

2. When R=2r, $\theta = \pi$ ，then H=0

§ geodesics on a torus

$$\begin{cases} x = c + a\cos v\cos u \\ y = c + a\cos v\sin u \\ z = a\sin v \end{cases}$$

By geodesic equations $\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k = 0$ we get

$$\ddot{u} - \frac{2a\sin v}{c + a\cos v}uv = 0$$

$$\ddot{v} + \frac{1}{a}\sin v c + a\cos v\dot{u}^2 = 0$$

Using the substituion w=c+acosv and integrating (with a trick or two)gives a solution in terms of u and v

$$\begin{cases} \dot{u} = \dfrac{k}{c + a\cos v^2} \\ \dot{v} = \pm\sqrt{-\dfrac{k^2}{a^2 c + a\cos v^2} + l} \end{cases}$$

The Clairaut parameterization of a torus treats it as a surface of revolution。

From it，we obtain a formula for $\dfrac{du}{dv}$

$$\frac{du}{dv} = \pm\frac{ah}{c + a\cos v\sqrt{c + a\cos v^2 - h^2}}$$

$$h = c + a\cos v\sin\varphi$$

The five families of geodesics：

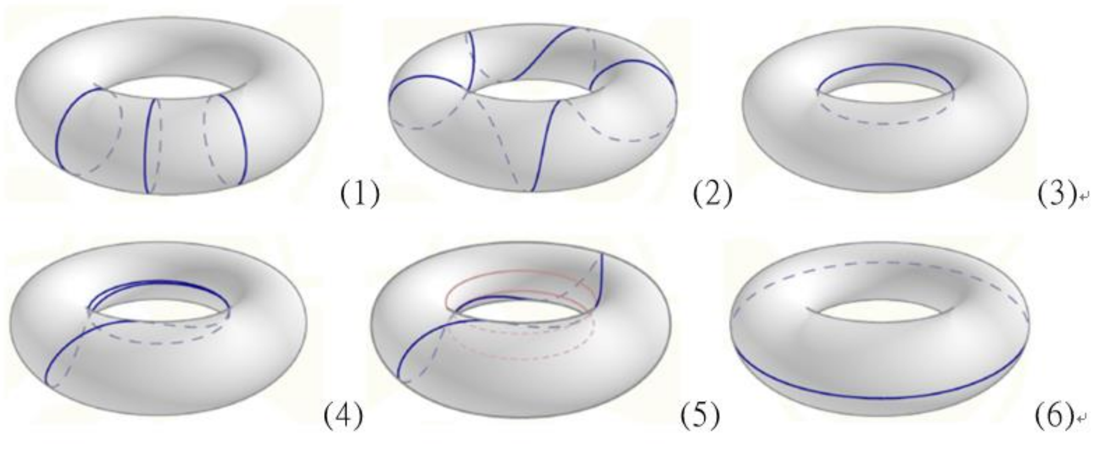
(1) h=0，these are the meridians

(2) 0<h<c-a unbounded geodesics，which alternately cross inner and outer equators。

(3) h=c-a the inner equator ，and geodesics asymptotic to it

(4) c-a<h<c+a bounded geodesics, which cross the outer equator but bounce off barrier curves

(5) h=c-a the outer equator



(1) (2) (3)

(4) (5) (6)

When h>c+a，there are no real solutions to the formula for $\frac{du}{dv}$